Sovereign Debt Ratchets and Welfare Destruction
Online Appendix

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A.1 Risk Premia

Debt investors' discounted cumulative gain can be expressed as follows:

$$\int_0^t e^{-\int_0^u (r(s_v)+m)dv} (\kappa + m)\, du + e^{-\int_0^t (r(s_u)+m)du} D\left(Y_t, F_t, s_t\right).$$

Since the discounted cumulative gain must be a $\mathbb{Q}$-martingale, we must have:

$$(r+m)\, D = \kappa + m + (\boldsymbol{\mu}_X - \boldsymbol{\sigma}_X \boldsymbol{\nu}) \cdot \partial_X D + \frac{1}{2}\mathrm{tr}\left(\boldsymbol{\sigma}_X' \partial_{XX'} D \boldsymbol{\sigma}_X\right) + (G - mF)\, \partial_F D.$$

The excess return on holding sovereign bonds between t and $t + dt$ must reflect price changes dD_t, coupon and principal payments $(\kappa + m)dt$, as well as reinvestment costs $mD_t dt$. In other words, those excess returns can be computed as follows:

$$dR_t - r_t dt = \frac{dD_t + (\kappa + m)dt - mD_t dt}{D_t} - r_t dt.$$

We use Itô formula to compute dD_t as follows:

$$dD_t = \left[\boldsymbol{\mu}_{X,t} \cdot \partial_X D_t + \frac{1}{2}\mathrm{tr}\left(\boldsymbol{\sigma}_{X,t}' \partial_{XX'} D_t \boldsymbol{\sigma}_{X,t}\right) + (G_t - mF_t)\, \partial_F D_t\right] dt + \left(\boldsymbol{\sigma}_{X,t}' \partial_X D_t\right) \cdot d\boldsymbol{B}_t. \quad (1)$$

Reinjecting equation (1) into our equation for excess returns, and using our martingale condition for D, we obtain the following formula for excess returns:

$$dR_t - r_t dt = \left(\boldsymbol{\sigma}_{X,t}' \partial_X \ln D_t\right) \cdot \boldsymbol{\nu}_t dt + \left(\boldsymbol{\sigma}_{X,t}' \partial_X \ln D_t\right) \cdot d\boldsymbol{B}_t.$$

The second term has zero conditional expectations (under $\mathbb{P}$), which leads to the following formula for expected excess returns once we define $\pi\left(Y, F, s\right) := \left(\boldsymbol{\sigma}_X \boldsymbol{\nu}\right) \cdot \partial_X \ln D$:

$$\mathbb{E}_t\left[dR_t - r_t dt\right] = \pi\left(Y_t, F_t, s_t\right) dt.$$

□

A.2 Citizens vs. Government

Assume that the citizens of the small open economy have linear preferences with discount rate $\hat{\delta} < \delta$, where δ is the discount rate of the government. In Section A.2.1, we treat the case where income is an arbitrary Itô process and upon default, the government looses its entire income stream and creditors suffer a full loss on their investment. In Section A.2.2 instead, we treat the case discussed in Section 5 of the main text: income follows geometric Brownian motion dynamics and upon default, income jumps down by a factor α and the debt suffers a haircut parameterized by θ.

A.2.1 General Case – Full Loss Upon Default

Assume for simplicity that upon default, the small open economy's income is zero forever after, and that creditors lose their entire investment. Let $C(Y,F,s)$ be the resulting country's consumption in state (Y,F,s) resulting from the government optimization outcome. Let V (resp. $\hat{V}$) be the present value of life-time utility for the government (resp. citizens consuming according to the government policy), and let $\hat{V}_0$ be the life-time utility of citizens of a country that no longer trades in financial markets, and which defaults according to the government default stopping rule. We then have:

$$V(Y,F,S) := \mathbb{E}_{Y,F,s}\left[\int_0^\tau e^{-\delta t} C(Y_t,F_t,s_t)dt\right]$$
$$\hat{V}(Y,F,S) := \mathbb{E}_{Y,F,s}\left[\int_0^\tau e^{-\hat{\delta} t} C(Y_t,F_t,s_t)dt\right]$$
$$\hat{V}_0(Y,F,S) := \mathbb{E}_{Y,F,s}\left[\int_0^\tau e^{-\hat{\delta} t}\left(Y_t-(\kappa+m)Fe^{-mt}\right)dt\right].$$

The equation defining $\hat{V}_0$ reflects the fact when the country refrains from future trading in international credit markets, the existing stock of sovereign debt amortizes at rate m. In all these value functions, the stopping time τ is the same, and is pinned down by the government's optimal behavior.

Our proof strategy will rely on the following key insight. While the government equalizes the marginal benefit of debt issuance D with the marginal cost $-\partial_F V$, the citizens of the small open economy–who are more patient than the government–will be worse off, as the marginal cost $-\partial_F \hat{V}$ from the perspective of citizens is greater than the marginal benefit D. We then show that the value differential $\hat{V}-\hat{V}_0$ is the expected net present value (at rate $\hat{\delta}$) of the product of (a) the utility losses $D+\partial_F\hat{V}$, times (b) the issuance rate G. Since the issuance rate is positive and the utility losses are negative, this will allow us to conclude that the value differential $\hat{V}-\hat{V}_0$ is always negative.

The value function for the government satisfies:

$$\delta V = C + \boldsymbol{\mu}_X\cdot\partial_X V + \frac{1}{2}\mathrm{tr}\left(\boldsymbol{\sigma}_X'\partial_{XX'}V\boldsymbol{\sigma}_X\right) + (G-mF)\,\partial_F V. \tag{2}$$

Using $\partial_F V + D = 0$, and $C(Y,F,s) = Y + G(Y,F,s)D(Y,F,s) - (\kappa+m)F$, we differentiate both side of equation (2) by F and show that the debt price satisfies:

$$(\delta+m)\,D = (\kappa+m) + \boldsymbol{\mu}_X\cdot\partial_X D + \frac{1}{2}\mathrm{tr}\left(\boldsymbol{\sigma}_X'\partial_{XX'}D\boldsymbol{\sigma}_X\right) - mF\partial_F D. \tag{3}$$

On the other hand, the value function $\hat{V}$ for citizens (who use a discount rate $\hat{\delta}<\delta$ and who consume according to the government policy) satisfies:

$$\hat{\delta}\hat{V} = C + \boldsymbol{\mu}_X\cdot\partial_X \hat{V} + \frac{1}{2}\mathrm{tr}\left(\boldsymbol{\sigma}_X'\partial_{XX'}\hat{V}\boldsymbol{\sigma}_X\right) + (G-mF)\,\partial_F \hat{V}. \tag{4}$$

Our next lemma shows that we must have $\hat{V}\geq V$.

Lemma 1 *For all (Y,F,s), it must be the case that $\hat{V}(Y,F,s)\geq V(Y,F,s)$.*

To prove Lemma 1, we subtract equation (2) from equation (4), and we use the boundary condition

$\hat{V}(Y,F,s) = V(Y,F,s) = 0$ at default time τ. We obtain:

$$\hat{V}(Y,F,S) - V(Y,F,S) = \mathbb{E}_{Y,F,s}\left[\int_0^\tau e^{-\hat{\delta}t}\left(\delta - \hat{\delta}\right)V(Y_t,F_t,s_t)dt\right] \geq 0.$$

The last inequality follows from the fact that we must have $V(Y,F,s) \geq 0$, since one feasible strategy for the government is to default immediately and obtain 0. Thus $\hat{V} \geq V$ in the interior of the continuation region (with equality on the default boundary). □

We then study the sign of $H := D + \partial_F \hat{V}$; this function represents the marginal value of one extra unit of debt, from the point of view of the patient citizens. We can prove the following:

Lemma 2 *For all (Y,F,s), it must be the case that $H(Y,F,s) < 0$ on the interior of the continuation region.*

To prove our lemma, we differentiate equation (4) w.r.t. F:

$$\begin{aligned}\left(\hat{\delta}+m\right)\partial_F\hat{V} = -(\kappa+m) + \left(D+\partial_F\hat{V}\right)\partial_F G + G\partial_F\left(D+\partial_F\hat{V}\right) + \boldsymbol{\mu}_X\cdot\partial_X\left(\partial_F\hat{V}\right)\\ + \frac{1}{2}\mathrm{tr}\left(\boldsymbol{\sigma}_X'\partial_{XX'}\left(\partial_F\hat{V}\right)\boldsymbol{\sigma}_X\right) - mF\partial_F\left(\partial_F\hat{V}\right). \quad (5)\end{aligned}$$

We add up equation (3) to equation (5), and leverage the definition of H to establish that it satisfies:

$$\left(\hat{\delta}+m-\partial_F G\right)H = (\hat{\delta}-\delta)D + \boldsymbol{\mu}_X\cdot\partial_X H + \frac{1}{2}\mathrm{tr}\left(\boldsymbol{\sigma}_X'\partial_{XX'}H\boldsymbol{\sigma}_X\right) + (G-mF)\,\partial_F H. \quad (6)$$

This implies that H admits an integral representation:

$$H(Y,F,s) = \mathbb{E}_{Y,F,s}\left[\int_0^\tau e^{-\int_0^t\left(\hat{\delta}+m-\partial_F G_u\right)du}(\hat{\delta}-\delta)D_t dt + e^{-\int_0^\tau\left(\hat{\delta}+m-\partial_F G_u\right)du}H(Y_\tau,F_\tau,s_\tau)\right]. \quad (7)$$

In the above, we have used the "short" notation $D_t = D\left(Y_t,F_t,s_t\right)$ and $G_t = G\left(Y_t,F_t,s_t\right)$. At the default boundary $(Y,F) \in \partial\mathcal{O}(s)$, the full loss given default implies that

$$\hat{V}\left(Y,F,s\right) = 0 \qquad D\left(Y,F,s\right) = 0.$$

Since Lemma 1 shows that $\hat{V}\left(Y,F,s\right) \geq V\left(Y,F,s\right) \geq 0$ in the continuation region, it means that at the default boundary $(Y,F) \in \partial\mathcal{O}(s)$, we must have $\partial_F\hat{V} \leq 0$. In other words, for $(Y,F) \in \partial\mathcal{O}(s)$, we have

$$H(Y,F,s) \leq 0. \quad (8)$$

The desired result of $H(Y,F,s) < 0$ then follows from the integral representation (7), $\hat{\delta} < \delta$, debt price D being strictly positive on the interior of the continuation region, and (8). □

Finally, we show that $\hat{V}(Y,F,s) < \hat{V}_0(Y,F,s)$ for all (Y,F,s) in the interior of the continuation region. We first write down the PDEs statisfied by $\hat{V}$ and $\hat{V}_0$:

$$\hat{\delta}\hat{V} = Y - (\kappa+m)F + GH + \boldsymbol{\mu}_X\cdot\partial_X\hat{V} + \frac{1}{2}\mathrm{tr}\left(\boldsymbol{\sigma}_X'\partial_{XX'}\hat{V}\boldsymbol{\sigma}_X\right) - mF\partial_F\hat{V} \quad (9)$$

$$\hat{\delta}\hat{V}_0 = Y - (\kappa+m)F + \boldsymbol{\mu}_X \cdot \partial_X \hat{V}_0 + \frac{1}{2}\text{tr}\left(\boldsymbol{\sigma}'_X \partial_{XX'}\hat{V}_0 \boldsymbol{\sigma}_X\right) - mF\partial_F \hat{V}_0. \tag{10}$$

Note $\Delta\hat{V} := \hat{V} - \hat{V}_0$, which then satisfies the following PDE:

$$\hat{\delta}\Delta\hat{V} = GH + \boldsymbol{\mu}_X \cdot \partial_X \Delta\hat{V} + \frac{1}{2}\text{tr}\left(\boldsymbol{\sigma}'_X \partial_{XX'}\Delta\hat{V} \boldsymbol{\sigma}_X\right) - mF\partial_F \Delta\hat{V} \tag{11}$$

At the default boundary it is immediate that $\Delta\hat{V}(Y,F,s) = 0$ (recall that the full loss given default implies that for $(Y,F) \in \partial\mathcal{O}(s)$ we have $\hat{V}(Y,F,s) = 0$ and $\hat{V}_0(Y,F,s) = 0$). Therefore $\hat{\Delta}V(Y,F,s)$ admits the following integral representation:

$$\Delta\hat{V}(Y,F,s) = \mathbb{E}^{nt}_{Y,F,S}\left[\int_0^\tau e^{-\hat{\delta}t} G_t H_t dt\right],$$

where the operator $\mathbb{E}^{nt}$ represents expectations under the no-trade policy. Since $G_t \geq 0$ almost surely, and since we have also proved that $H_t \leq 0$ almost surely, the desired result of $\hat{V}(Y,F,s) < \hat{V}_0(Y,F,s)$ follows. □

A.2.2 Particular Case – Geometric Brownian Motion Income and Reinjections

We now tackle the case where income at default drops from $Y_{\tau-}$ to $Y_\tau = \alpha Y_{\tau-}$, where the debt suffers a haircut $1-\alpha\theta$, and where the aggregate state s_t is trivially equal to 1. When income follows geometric Brownian motion dynamics, we provide in Section 5 of the main text a complete analytical characterization of the value function, debt prices and default boundary. In particular, we show that our Smooth MPE exists for any $\delta > r$, and that the default boundary $\bar{x}_\delta$ is a decreasing function of the impatience rate δ. We begin by showing the following lemma.

Lemma 3 *For fixed x, the debt price function is decreasing in the parameter δ.*

In order to prove our assertion, remember that the debt price d satisfies the following:

$$\begin{aligned}(\delta+m)d(x;\delta) &= \kappa + m - (\mu + m - |\boldsymbol{\sigma}|^2)x\partial_x d(x;\delta) + \frac{1}{2}|\boldsymbol{\sigma}|^2 x^2 \partial_{xx} d(x;\delta) \\ d(\bar{x};\delta) &= \alpha\theta d(\theta\bar{x};\delta).\end{aligned}$$

In the above, we have used a notation that emphasizes that the debt price depends on the parameter δ. Differentiate these equations above w.r.t δ to obtain:

$$(\delta+m)\partial_\delta d(x;\delta) = -d(x;\delta) - (\mu+m-|\boldsymbol{\sigma}|^2)x\partial_{\delta x} d(x;\delta) + \frac{1}{2}|\boldsymbol{\sigma}|^2 x^2 \partial_{\delta xx} d(x;\delta) \tag{12}$$

$$\frac{\partial\bar{x}}{\partial\delta}\partial_x d(\bar{x};\delta) + \partial_\delta d(\bar{x};\delta) = \alpha\theta^2 \frac{\partial\bar{x}}{\partial\delta}\partial_x d(\theta\bar{x};\delta) + \alpha\theta\partial_\delta d(\theta\bar{x};\delta). \tag{13}$$

Using the expression for d established in Section 5 of the main text and some algebra, we compute:

$$\partial_x d(\bar{x};\delta) = -\left(\frac{\xi-1}{\bar{x}}\right)\left(\frac{\kappa+m}{\delta+m}\right)\left(\frac{1-\alpha\theta}{1-\alpha\theta^\xi}\right)$$

$$\partial_x d\left(\theta\bar{x};\delta\right) = -\theta^{\xi-2}\left(\frac{\xi-1}{\bar{x}}\right)\left(\frac{\kappa+m}{\delta+m}\right)\left(\frac{1-\alpha\theta}{1-\alpha\theta^{\xi}}\right) = \theta^{\xi-2}\partial_x d\left(\bar{x};\delta\right).$$

Thus, equation (13) can be re-written:

$$\partial_\delta d\left(\bar{x};\delta\right) = \alpha\theta\partial_\delta d\left(\theta\bar{x};\delta\right) + \frac{\partial\bar{x}}{\partial\delta}\partial_x d\left(\bar{x};\delta\right)\left(\alpha\theta^{\xi}-1\right). \tag{14}$$

The last term on the right handside of equation (14) is always negative, since $\partial\bar{x}/\partial\delta < 0$, since $\partial_x d < 0$ and since $\alpha\theta^{\xi} < 1$. Equation (12) admits a "source" term $-d(x;\delta)$, while equation (14) is the boundary condition for a function whose payoff is $\frac{\partial\bar{x}}{\partial\delta}\partial_x d\left(\bar{x};\delta\right)\left(1-\alpha\theta^{\xi}\right)$ each time the barrier $\bar{x}$ is hit, with discount factor $\alpha\theta$. Thus, equations (12) and (14) taken together allow us to write the following integral representation for $\partial_\delta d$:

$$\partial_\delta d(x;\delta) = -\mathbb{E}_x^{nt}\left[\int_0^\infty (\alpha\theta)^{N_{d,t}^{(\tau)}} e^{-(\delta+m)t} d\left(x_t;\delta\right)dt + \sum_{k=1}^{\infty} e^{-(\delta+m)\tau_k}(\alpha\theta)^{k+1}\frac{\partial\bar{x}}{\partial\delta}\partial_x d\left(\bar{x};\delta\right)\left(1-\alpha\theta^{\xi}\right)\right] < 0,$$

where $\mathbb{E}_x^{nt}$ is the expectation operator under the no-trade policy, in other words under which x_t evolves according to:

$$dx_t = -(m+\mu-|\boldsymbol{\sigma}|^2)x_t dt - x_t\boldsymbol{\sigma}\cdot d\boldsymbol{B}_t + (\theta-1)\,\bar{x}dN_{d,t}.$$

Thus, we have showed that $D_t = d(x_t;\delta)$ is a decreasing function of δ, for fixed x. □

Armed with Lemma 3, we can prove our main assertion. Introduce the following notation:

$$V_0\left(Y,F;\delta\right) := \sup_{\boldsymbol{\tau}}\mathbb{E}_{Y,F}\left[\int_0^{+\infty} e^{-\delta t}\left(\alpha^{N_t^{(\tau)}}Y_t - (\alpha\theta)^{N_t^{(\tau)}}e^{-mt}(\kappa+m)F\right)dt\right].$$

In other words, $V_0\left(Y,F;\delta\right)$ equals the no-trade value, using discount rate δ, when the borrower has an option to default at any point time. Let $N_{\delta,t}^*$ be the counting process associated with the sequence $\boldsymbol{\tau}_\delta^*$ of optimal default times that solves the optimal stopping problem above. Note D_δ the debt price and G_δ^* the optimal issuance rate of the related Smooth MPE, in which the borrower discounts at rate δ:

$$D_\delta\left(Y,F\right) = -\partial_F V_0\left(Y,F;\delta\right)$$
$$G_\delta^*\left(Y,F\right) = (\delta-r)\frac{\partial_F V_0\left(Y,F;\delta\right)}{-\partial_{FF}V_0\left(Y,F;\delta\right)}.$$

Note $C_\delta^*(Y,F)$ the equilibrium consumption policy when the borrower discounts at rate δ:

$$C_\delta^*(Y,F) := Y + G_\delta^*(Y,F)D_\delta(Y,F) - (\kappa+m)F.$$

Take $\hat{\delta} < \delta$, and consider $V_0\left(Y,F;\hat{\delta}\right)$, the equilibrium value function for a borrower with discount rate $\hat{\delta}$; remember it is also equal to the no trade value. Using Lemma 3, D_δ is decreasing in δ, and we can thus write the following:

$$\begin{aligned}
V_0\left(Y,F;\hat{\delta}\right) &= \sup_{\tau,G} \mathbb{E}_{Y,F}\left[\int_0^{+\infty} e^{-\hat{\delta}t}\left(Y_t^{(\tau)} + G_t D_{\hat{\delta}}\left(Y_t^{(\tau)}, F_t^{(\tau,G)}\right) - (\kappa+m)\, F_t^{(\tau,G)}\right) dt\right] \\
&\geq \mathbb{E}_{Y,F}\left[\int_0^{+\infty} e^{-\hat{\delta}t}\left(Y_t^{(\tau_\delta^*)} + G_\delta^*\left(Y_t^{(\tau_\delta^*)}, F_t^{(\tau_\delta^*,G_\delta^*)}\right) D_{\hat{\delta}}\left(Y_t^{(\tau_\delta^*)}, F_t^{(\tau_\delta^*,G_\delta^*)}\right) - (\kappa+m)\, F_t^{(\tau_\delta^*,G_\delta^*)}\right) dt\right] \\
&\geq \mathbb{E}_{Y,F}\left[\int_0^{+\infty} e^{-\hat{\delta}t}\left(Y_t^{(\tau_\delta^*)} + G_\delta^*\left(Y_t^{(\tau_\delta^*)}, F_t^{(\tau_\delta^*,G_\delta^*)}\right) D_{\delta}\left(Y_t^{(\tau_\delta^*)}, F_t^{(\tau_\delta^*,G_\delta^*)}\right) - (\kappa+m)\, F_t^{(\tau_\delta^*,G_\delta^*)}\right) dt\right] \\
&= \mathbb{E}_{Y,F}\left[\int_0^{+\infty} e^{-\hat{\delta}t} C_\delta^*\left(Y_t^{(\tau_\delta^*)}, F_t^{(\tau_\delta^*,G_\delta^*)}\right) dt\right].
\end{aligned}$$

The first inequality is due to the fact that the financing policy G_δ^* and the default policy $\boldsymbol{\tau}_\delta^*$ are feasible but not necessarily optimal for a borrower discounting at rate $\hat{\delta}$. The second inequality is due to the fact that the equilibrium bond price $D_{\hat{\delta}}$ is greater than the equilibrium bond price D_δ. Thus, we have proven that the equilibrium value $V_{\hat{\delta}}$ (which equals the corresponding no-trade value) is greater than the indirect utility function $\mathbb{E}_{Y,F}\left[\int_0^{+\infty} e^{-\hat{\delta}t} C_\delta^*\left(Y_t^{(\tau_\delta^*)}, F_t^{(\tau_\delta^*,G_\delta^*)}\right) dt\right]$, which represents the welfare for citizens (discount at rate $\hat{\delta}$) of a country whose government discounts at rate $\delta > \hat{\delta}$. □

A.3 Transaction Costs and Non-Pecuniary Benefits

As discussed in the main text, our results can be extended to the case where the government incurs transaction costs upon the issuance of bonds, or when the government enjoys non-pecuniary benefits, at the cost of Assumption 1 described below. For simplicity, we only cover the case with transaction costs – since the modeling approach is identical to the case with non-pecuniary benefits.

For ease of exposition, assume that the exogenous SDF state s_t is trivially equal to 1, and that $\boldsymbol{\nu}(s) = 0$ always (so creditors and the government discount cashflows under the same probability measure). Consider an environment where the government incurs, when issuing bonds, (i) a proportional cost $\eta \in (0,1)$ on total proceeds raised from issuance, and (ii) a proportional cost $b > 0$ on the notional amount of bonds issued. In particular, assume that those costs are borne only when $G_t \geq 0$ – in other words we assume throughout that if $G_t < 0$, the government does not get a rebate for buying back existing debt. The case of the government enjoying non-pecuniary benefits over debt issuance corresponds to $\eta < 0$ and $b < 0$.

Assumption 1 *There exists a Smooth MPE without transaction costs, and the no-trade value function $V_0(\cdot,\cdot)$ satisfies, for any (Y,F) in the continuation region:*

$$-\partial_F V_0(Y,F) \geq \frac{1}{\delta - r}\left[\frac{\eta\,(\kappa+m)}{1-\eta} + \frac{(\delta+m)\, b}{1-\eta}\right]. \tag{15}$$

Assumption 1 can be verified without solving the equilibrium with transaction cost. As we will see, this restriction guarantees that the issuance rate in the economy with transaction costs stays weakly positive, so that an optimality condition similar to equation (22) of the main text is always satisfied.

In the setting with transaction costs, in a Smooth MPE, the consumption rate enjoyed by the government is equal to:

$$C_t = Y_t - (\kappa + m)F_t + (1 - \eta 1_{\{G_t \geq 0\}})G_t D_t - b 1_{\{G_t \geq 0\}} G_t.$$

Postulate that a Smooth MPE exists with transaction costs, and note V_{tc}, D_{tc} the corresponding value function and debt prices; whenever $G_t \geq 0$, a necessary condition for optimality is

$$(1 - \eta)D_{tc} - b + \partial_F V_{tc} = 0. \tag{16}$$

Once we reinject this first order condition into the HJB equation satisfied by V_{tc}, it becomes straightforward to notice that in a Smooth MPE with transaction costs, the government welfare must equal its no-trade value: $V_{tc} = V_0$. The default policy of the government is identical in both economic environments. The debt price D_{tc} can be derived from equation (16) while the optimal bond issuance policy G_{tc} can be computed by using the asset pricing equation satisfied by D_{tc}:

$$\begin{aligned} D_{tc}(Y, F) &= \frac{b - \partial_F V_0(Y, F)}{1 - \eta} \\ G^*_{tc}(Y, F) &= \frac{(\delta - r)\, D_{tc}(Y, F) - \frac{\eta}{1-\eta}\,(\kappa + m) - \frac{(\delta+m)b}{1-\eta}}{-\partial_F D_{tc}(Y, F)}. \end{aligned}$$

Assumption 1 then guarantees that $G^*_{tc}(Y, F) \geq 0$, which allows us to confirm the validity of the first order condition (16) across the state space. The above reasoning then shows that we can construct a Smooth MPE with transaction costs, so long as the Smooth MPE without transaction costs exists and so long as the restriction of Assumption 1 is satisfied. The expressions for D_{tc} and G_{tc} then makes it clear that transaction costs decrease the pace of bond issuances, and increase the pricing of debt. □

B Proofs for: Geometric Brownian Motion

B.1 Smooth MPE

For a given admissible default policy $\boldsymbol{\tau} \in \mathcal{T}$, define $N_{d,t}^{(\boldsymbol{\tau})} := \max\{k \in \mathbb{N} : \tau_k \leq t\}$ to be the counting process for default events. Using this notation, the dynamic evolution of the controlled stochastic process $Y_t^{(\boldsymbol{\tau})}$ can be expressed as follows:

$$Y_t^{(\boldsymbol{\tau})} = \alpha^{N_{d,t}^{(\boldsymbol{\tau})}} Y_t.$$

Similarly, the dynamic evolution of the controlled stochastic process $F^{(G,\boldsymbol{\tau})}$ can be expressed as follows:

$$F_t^{(G,\boldsymbol{\tau})} = \int_0^t \left(G\left(Y_u^{(\boldsymbol{\tau})}, F_u^{(G,\boldsymbol{\tau})}, s_u\right) - mF_u^{(G,\boldsymbol{\tau})}\right) du + \int_0^t (\theta\alpha - 1) F_{u-}^{(G,\boldsymbol{\tau})} dN_{d,u}^{(\boldsymbol{\tau})}.$$

Armed with those notations, notice that V can be written as follows:

$$\begin{aligned} V(Y,F,s) &= \sup_{(G,\boldsymbol{\tau})\in\mathcal{G}\times\mathcal{T}} \mathbb{E}_{Y,F,s}\left[\int_0^{+\infty} e^{-\delta t}\left(Y_t^{(\boldsymbol{\tau})} + G\left(Y_t^{(\boldsymbol{\tau})}, F_t^{(G,\boldsymbol{\tau})}, s_t\right) D_t - (\kappa+m) F_t^{(G,\boldsymbol{\tau})}\right) dt\right] \\ &= Y \sup_{(g,\boldsymbol{\tau})\in\mathcal{G}\times\mathcal{T}} \mathbb{E}_{x,s}\left[\int_0^{+\infty} \alpha^{N_{d,t}^{\tau}} e^{-\left(\delta-\mu+\frac{\sigma^2}{2}\right)t+\sigma B_t}\left(1 + g\left(x_t^{(g,\boldsymbol{\tau})}, s_t\right) D_t - (\kappa+m) x_t^{(g,\boldsymbol{\tau})}\right) dt\right]. \end{aligned}$$

On $(0,\bar{x})$, the debt-to-income ratio $x_t^{(g,\boldsymbol{\tau})}$ is a controlled stochastic process that evolves as follows:

$$dx_t^{(g,\boldsymbol{\tau})} = \left(g(x_t^{(g,\boldsymbol{\tau})}, s_t) - \left(m + \mu - |\boldsymbol{\sigma}|^2\right) x_t^{(g,\boldsymbol{\tau})}\right) dt - x_t^{(g,\boldsymbol{\tau})} \boldsymbol{\sigma} \cdot d\boldsymbol{B_t} + (\theta - 1)\, dN_{d,t}^{(\boldsymbol{\tau})}.$$

The scaled value function $v := \frac{V}{Y}$ is equal to:

$$v(x,s) = \sup_{(g,\boldsymbol{\tau})\in\mathcal{G}\times\mathcal{T}} \tilde{\mathbb{E}}_{x,s}\left[\int_0^{+\infty} \alpha^{N_{d,t}^{\tau}} e^{-(\delta-\mu)t}\left(1 + g\left(x_t^{(g,\boldsymbol{\tau})}, s_t\right) D_t - (\kappa+m) x_t^{(g,\boldsymbol{\tau})}\right) dt\right]. \tag{17}$$

In equation (17), we have introduced the measure $\tilde{\Pr}$, defined for any arbitrary Borel set $A \subseteq \mathcal{F}_t$ via $\tilde{\Pr}(A) = \mathbb{E}\left[\exp\left(-\frac{|\sigma|^2}{2}t + \boldsymbol{\sigma}\cdot\boldsymbol{B}_t\right) A\right]$. Under such measure, in the continuation region $(0,\bar{x})$, using Girsanov's theorem, the controlled debt-to-income ratio $x_t^{(g,\boldsymbol{\tau})}$ evolves as follows:

$$dx_t^{(g,\boldsymbol{\tau})} = \left(g(x_t^{(g,\boldsymbol{\tau})}, s_t) - (m + \mu)\, x_t^{(g,\boldsymbol{\tau})}\right) dt - x_t^{(g,\boldsymbol{\tau})} \boldsymbol{\sigma} \cdot d\tilde{\boldsymbol{B}}_{\boldsymbol{t}} + (\theta - 1)\, x_t^{(g,\boldsymbol{\tau})} dN_{d,t}^{(\boldsymbol{\tau})}. \tag{18}$$

$\tilde{\boldsymbol{B}}_{\boldsymbol{t}} := \boldsymbol{B_t} - \boldsymbol{\sigma} t$ is a standard Brownian motion under $\tilde{\Pr}$. As discussed in main text, the government welfare can be computed as if the government was never issuing debt – in that case, v is not dependent on the capital market conditions, and thus independent of the state variable s. Thus, for $x \in (0,\bar{x})$, v satisfies:

$$(\delta - \mu)\, v(x) = 1 - (\kappa + m)\, x - (\mu + m)\, x v'(x) + \frac{1}{2}|\boldsymbol{\sigma}|^2 x^2 v''(x). \tag{19}$$

This is a second order ordinary differential equation, whose general solutions are power functions of x. The exponent of the general solutions solves the quadratic equation:

$$\frac{1}{2}|\sigma|^2\xi^2 - \left(m + \mu + \frac{1}{2}|\sigma|^2\right)\xi - (\delta - \mu) = 0.$$

Given that $\delta > \mu$, this quadratic equation admits one positive, and one negative roots. Since v must be finite as $x \to 0$, we eliminate the negative root, and note $\xi > 1$ the positive one. We need one more boundary condition – we will use the fact that upon default at $x = \bar{x}$, the small open economy suffers a discrete income drop by a factor α, and immediately restructure its debt so that its post-default debt-to-income ratio is a fraction θ of its pre-default value:

$$v(\bar{x}) = \alpha v(\theta \bar{x}).$$

Using these, we can express v as follows on $[0, \bar{x}]$:

$$v(x) = \frac{1}{\delta - \mu}\left[1 - \left(\frac{1-\alpha}{1-\alpha\theta^{\xi}}\right)\left(\frac{x}{\bar{x}}\right)^{\xi}\right] - \left(\frac{\kappa + m}{\delta + m}x\right)\left[1 - \left(\frac{1-\alpha\theta}{1-\alpha\theta^{\xi}}\right)\left(\frac{x}{\bar{x}}\right)^{\xi-1}\right].$$

For $x > \bar{x}$, let $n(x;\bar{x}) := 1 + \lfloor \frac{\ln x - \ln \bar{x}}{-\ln \theta} \rfloor$ be the number of times the government needs to default consecutively in order to re-enter the continuation region. For $x > \bar{x}$, the value function satisfies:

$$v(x) = \alpha^{n(x;\bar{x})} v\left(\theta^{n(x;\bar{x})} x\right).$$

Finally, we "paste" the solution for $x > \bar{x}$ with the solution for $x \leq \bar{x}$, in such a way that the function v is $\mathcal{C}^1$ on $\mathbb{R}^+$, so that we can use standard verification arguments. The determination of $\bar{x}$ relies on the smooth pasting condition:

$$v'(\bar{x}) = \alpha\theta v'(\theta\bar{x}).$$

This leads to the following default boundary $\bar{x}$:

$$\bar{x} = \frac{\xi}{\xi - 1}\left(\frac{\delta + m}{\kappa + m}\right)\left(\frac{1-\alpha}{1-\alpha\theta}\right)\frac{1}{\delta - \mu}.$$

The debt price d per unit of debt outstanding can be computed by leveraging the fact that $d(x) = -v'(x)$. In other words, for $x \in [0, \bar{x}]$, we have:

$$d(x) = \left(\frac{\kappa + m}{\delta + m}\right)\left[1 - \left(\frac{1-\alpha\theta}{1-\alpha\theta^{\xi}}\right)\left(\frac{x}{\bar{x}}\right)^{\xi-1}\right].$$

For $x > \bar{x}$, d is determined via the number of consecutive times the sovereign will default in order to reenter the continuation region:

$$d(x) = (\alpha\theta)^{n(x;\bar{x})} d\left(\theta^{n(x;\bar{x})} x\right).$$

Note that in the continuation region, the value function v takes the following form:

$$v(x) = \frac{1}{\delta - \mu}\left(1 - \left(\frac{1-\alpha}{1-\alpha\theta^{\xi}}\right)\left(\frac{x}{\bar{x}}\right)^{\xi}\right) - xd(x).$$

This expression has an economic interpretation: the scaled value function v is equal to the no-trade value $1/(\delta - \mu)$, adjusted for the dead-weight costs of default, minus the scaled value of aggregate debt outstanding. The required expected excess return on the sovereign debt is:

$$\pi(x,s) = -\frac{xd'(x)}{d(x)}\boldsymbol{\sigma}\cdot\boldsymbol{\nu}(s) = \frac{\xi - 1}{\left(\frac{1-\alpha\theta^{\xi}}{1-\alpha\theta}\right)\left(\frac{\bar{x}}{x}\right)^{\xi-1} - 1}\boldsymbol{\sigma}\cdot\boldsymbol{\nu}(s),$$

which implies the equilibrium issuance policy:

$$g^*(x,s) = \frac{d(x)}{-d'(x)}(\delta - r(s) - \pi(s)) = \frac{\delta - r(s)}{\xi - 1}\left[\left(\frac{1-\alpha\theta^{\xi}}{1-\alpha\theta}\right)\left(\frac{\bar{x}}{x}\right)^{\xi-1} - 1\right]x - x\boldsymbol{\sigma}\cdot\boldsymbol{\nu}(s).$$

Finally, we need to establish that no other admissible policy can achieve a higher welfare for the government, via a standard verification theorem. Let $(g,\boldsymbol{\tau}) \in \mathcal{G}\times\mathcal{T}$ be an arbitrary issuance and default policy. We introduce the infinitessimal generator $\mathcal{L}^{(g)}$, defined for any function $f \in \mathcal{C}^2(\mathbb{R})$ as follows:

$$\mathcal{L}^{(g)}f(x) := (g(x,s) - (\mu + m)x)\,f'(x) + \frac{1}{2}x^2|\boldsymbol{\sigma}|^2 f''(x).$$

Note that the function v constructed above is defined on $\mathbb{R}_+$, and is $\mathcal{C}^2$ on $\mathbb{R}\setminus\{\theta^k\bar{x}; k\in\mathbb{N}\}$. At $x = \theta^k\bar{x}$ ($k\in\mathbb{N}$), the function v is $\mathcal{C}^1$ by construction. The function v also satisfies the variational inequality:

$$0 = \max\left[\sup_g\left[-(\delta-\mu)v(x) + 1 + gd(x) - (\kappa+m)x + \mathcal{L}^{(g)}v(x)\right]; \alpha v(\theta x) - v(x)\right]. \tag{20}$$

Assume $x_0 = x$; given the dynamic evolution of the controlled stochastic process $x_t^{(g,\boldsymbol{\tau})}$ (as described by equation (18)), we have the following Itô-Tanaka-Meyer formula:

$$\alpha^{N_{d,t}^{(\tau)}}e^{-(\delta-\mu)t}v(x_t^{(g,\boldsymbol{\tau})}) = v(x) - \int_0^t \alpha^{N_{d,u}^{(\tau)}}e^{-(\delta-\mu)u}x_u^{(g,\boldsymbol{\tau})}v'(x_u^{(g,\boldsymbol{\tau})})\boldsymbol{\sigma}\cdot d\tilde{\boldsymbol{B}}_{\boldsymbol{u}}$$
$$+\int_0^t \alpha^{N_{d,u}^{(\tau)}}e^{-(\delta-\mu)u}\left[\mathcal{L}^{(g)}v(x_u^{(g,\boldsymbol{\tau})}) - (\delta-\mu)v(x_u^{(g,\boldsymbol{\tau})})\right]du + \int_0^t \alpha^{N_{d,u-}^{(\tau)}}e^{-(\delta-\mu)u}\left[\alpha v(\theta x_{u-}^{(g,\boldsymbol{\tau})}) - v(x_{u-}^{(g,\boldsymbol{\tau})})\right]dN_{d,u}^{(\tau)}.$$

See for example Protter (2005). We then use our variational inequality (20) to obtain:

$$\alpha^{N_{d,t}^{(\tau)}}e^{-(\delta-\mu)t}v(x_t^{(g,\boldsymbol{\tau})}) \le v(x) - \int_0^t \alpha^{N_{d,u}^{\tau}}e^{-(\delta-\mu)u}\left[1 + g(x_u^{(g,\boldsymbol{\tau})}, s_u) - (\kappa+m)x_u^{(g,\boldsymbol{\tau})}\right]du$$
$$- \int_0^t \alpha^{N_{d,u}^{(\tau)}}e^{-(\delta-\mu)u}x_u^{(g,\boldsymbol{\tau})}v'(x_u^{(g,\boldsymbol{\tau})})\boldsymbol{\sigma}\cdot d\tilde{\boldsymbol{B}}_{\boldsymbol{u}}.$$

The stochastic integral in the second line of the equation above is a martingale since $xv'(x)$ is bounded.

Thus, taking expectations on both sides of this equality, we obtain:

$$\tilde{\mathbb{E}}_{x,s}\left[\int_0^t \alpha^{N^{(\tau)}_{d,u}} e^{-(\delta-\mu)u}\left[1+g(x_u^{(g)},s_u)-(\kappa+m)x_u^{(g)}\right]du + \alpha^{N^{(\tau)}_{d,t}} e^{-(\delta-\mu)t} v(x_t)\right] \le v(x).$$

When we take $t \to +\infty$, $\alpha^{N^{(\tau)}_{d,t}} e^{-(\delta-\mu)t} v(x_t) \to 0$. Using the dominated convergence theorem, we then obtain the desired result: for any admissible policy $(g, \boldsymbol{\tau})$, we have

$$v(x) \ge \tilde{\mathbb{E}}_{x,s}\left[\int_0^{+\infty} \alpha^{N^{\tau}_{d,t}} e^{-(\delta-\mu)t}\left(1+g\left(x_t^{(g,\boldsymbol{\tau})},s_t\right)D_t-(\kappa+m)x_t^{(g,\boldsymbol{\tau})}\right)dt\right].$$

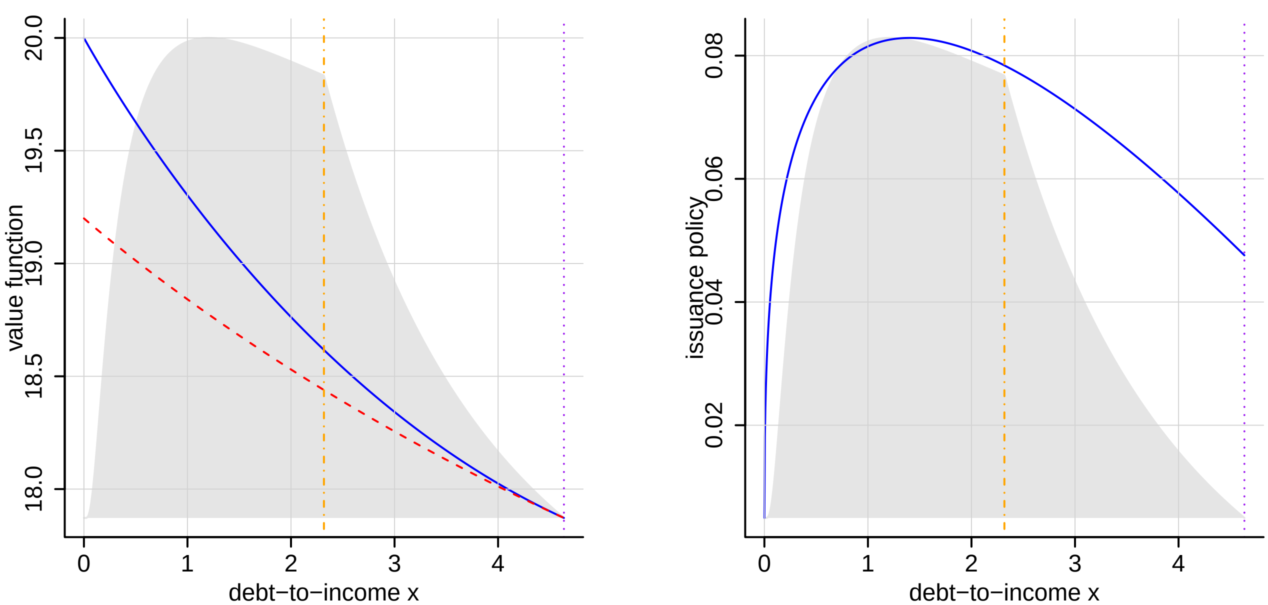
The bound is achieved for our issuance policy g^* and default policy $\boldsymbol{\tau}^*$, and the proof relies on steps identical to those described above, except that inequalities are now replaced by equalities. □

B.2 Equilibrium Illustration when $\xi \in (1, 2)$

In this section, we illustrate the Smooth MPE in the case where income follows geometric Brownian motion dynamics and the constant $\xi \in (1, 2)$. This corresponds to a parameter configuration as described in Lemma 3 of the main text. In that case, the debt price function is a convex (rather than concave) function of the debt-to-income ratio, and the issuance rate is hump-shaped, rather than being monotone decreasing in x.

Figure 1: Value Function and Issuance Policy

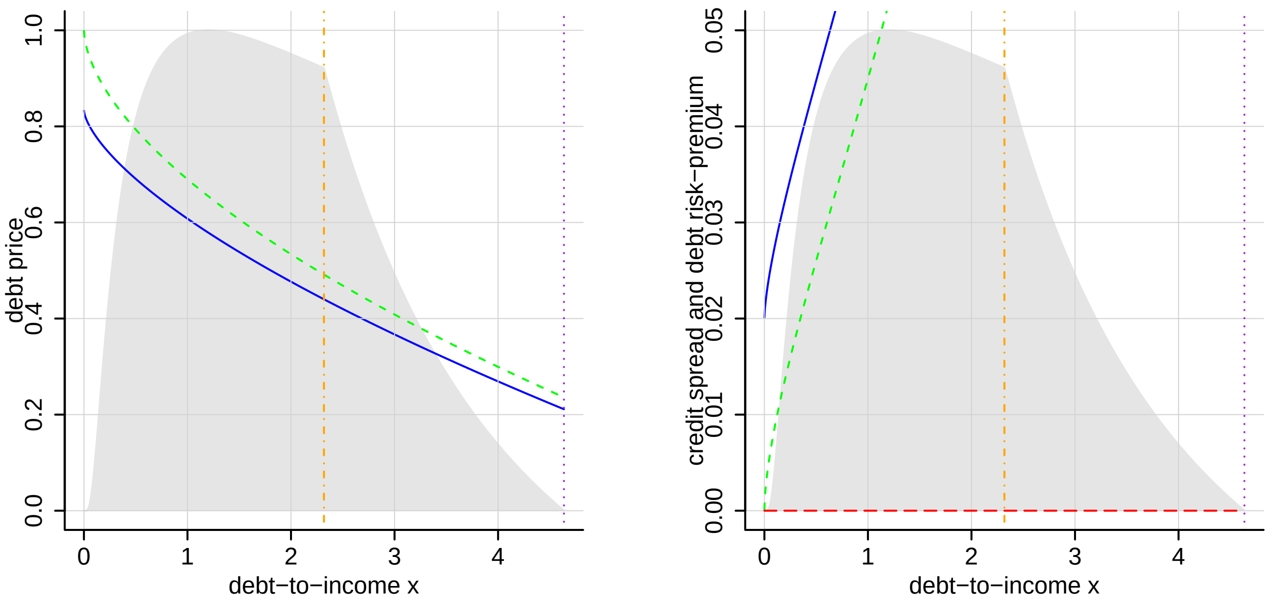
(a): Value Function $v(x)$ **(b):** Issuance Rate $g(x)$



In plot (a) the value function $v(x)$ is depicted in solid blue while the default value $\alpha v(\theta x)$ is depicted in dashed red. In both plots, the dotted purple vertical line is the default boundary, the dot-dash orange vertical line is the reinjection point, and the shaded grey area represents the stationary distribution of the debt-to-income ratio. The plots were computed assuming $\mu = 2\%$ p.a., $\sigma = 55\%$ p.a., $1/m = 20$ years, $\theta = 50\%$, $\alpha = 96\%$, $\nu = 0\%$, $r = \kappa = 5\%$ and $\delta = 7\%$.

Figure 2: Debt Price and Credit Spreads

(a): Debt price $d(x)$ **(b):** Credit spread $\varsigma(x)$



Solid blue line shows the no-commitment bond price and credit spread. Dash green line shows the bond price and credit spread in the corresponding model with no future issuance (or buybacks) of government debt. Long dash red line on the right hand side shows the bond risk-premium. In both plots, the dotted purple vertical line is the default boundary, the dot-dash orange vertical line is the reinjection point, and the shaded grey area represents the stationary distribution of the debt-to-income ratio in our Smooth MPE. Plots computed assuming $\mu = 2\%$ p.a., $\sigma = 55\%$ p.a., $1/m = 20$ years, $\theta = 50\%$, $\alpha = 96\%$, $\nu = 0\%$, $r = \kappa = 5\%$ and $\delta = 7\%$.

B.3 Attraction Point

Remember that the issuance policy is $G(Y, F) = Yg(x)$, with g that takes the following form:

$$g(x) = \frac{\delta - r}{\xi - 1}\left[\left(\frac{1 - \alpha\theta^{\xi}}{1 - \alpha\theta}\right)\left(\frac{\bar{x}}{x}\right)^{\xi - 1} - 1\right] x - \boldsymbol{\nu} \cdot \boldsymbol{\sigma} x.$$

Since $dF_t = (G(Y_t, F_t) - mF_t)\,dt$, the debt face value evolves as follows:

$$\frac{dF_t}{F_{t-}} = \left[\left(\frac{\delta - r}{\xi - 1}\right)\left(\frac{1 - \alpha\theta^{\xi}}{1 - \alpha\theta}\right)\bar{x}^{\xi - 1}\left(\frac{F_t}{Y_t}\right)^{1-\xi} - \left(\frac{\delta - r}{\xi - 1} + m + \boldsymbol{\nu} \cdot \boldsymbol{\sigma}\right)\right] dt - (1 - \alpha\theta)\,dN_{d,t}.$$

This allows us to compute the dynamics of $e^{\eta t} F_t^{\xi - 1}$:

$$\begin{aligned} d\left(e^{\eta t} F_t^{\xi-1}\right) &= \eta e^{\eta t}\left(x_a Y_t\right)^{\xi-1} dt - e^{\eta t}\left(1 - (\alpha\theta)^{\xi-1}\right)\left(\bar{x} Y_{t-}\right)^{\xi-1} dN_{d,t} \\ \Rightarrow F_t &= \left[e^{-\eta t} F_0^{\xi-1} + \eta \int_0^t e^{\eta(u-t)}\left(x_a Y_u\right)^{\xi-1} du - \left(1 - (\alpha\theta)^{\xi-1}\right)\int_0^t e^{\eta(u-t)}\left(\bar{x} Y_{u-}\right)^{\xi-1} dN_{d,u}\right]^{1/(\xi-1)}. \end{aligned}$$

The speed of mean-reversion η and the debt-to-income attraction point x_a are defined via

$$\eta := \delta - r + (\xi - 1)\,(m + \boldsymbol{\nu} \cdot \boldsymbol{\sigma})$$
$$x_a := \bar{x} \left[\left(\frac{1 - \alpha\theta}{1 - \alpha\theta^{\xi}} \right) \left(\frac{\xi - 1}{\delta - r} (m + \boldsymbol{\nu} \cdot \boldsymbol{\sigma}) + 1 \right) \right]^{\frac{1}{1-\xi}} .$$

□

B.4 Comparative Statics – Analytical Results

B.4.1 Default Boundary

Recall that the constant ξ is the positive root of the quadratic equation

$$\frac{1}{2}\sigma^2\xi^2 - \left(m + \mu + \frac{1}{2}\sigma^2 \right) \xi - (\delta - \mu) = 0.$$

Moreover, recall that $\xi > 1$. Finally, the default boundary, as showed in Section 5 of the main text, is

$$\bar{x} = \frac{\xi}{\xi - 1} \left(\frac{\delta + m}{\kappa + m} \right) \left(\frac{1 - \alpha}{1 - \alpha\theta} \right) \frac{1}{\delta - \mu}.$$

We now derive the comparative statics for $\bar{x}$.

- For the comparative static with respect to σ, since $\xi > 1$, note that

$$\frac{\partial \xi}{\partial \sigma^2} = \frac{-(\xi - 1)\xi^2}{\sigma^2\xi^2 + 2(\delta - \mu)} < 0.$$

  Since $\frac{\partial \bar{x}}{\partial \xi} < 0$, and since $\bar{x}$ does not depend directly on σ, it means that the default boundary $\bar{x}$ is increasing as output volatility σ increases.

- For the comparative static w.r.t. μ, notice that:

$$\frac{\partial \xi}{\partial \mu} = \frac{\xi(\xi - 1)}{\frac{\sigma^2}{2}\xi^2 + \delta - \mu} > 0.$$

  Thus, we can write:

$$\frac{d\bar{x}}{d\mu} = \frac{\partial \bar{x}}{\partial \mu} + \frac{\partial \bar{x}}{\partial \xi}\frac{\partial \xi}{\partial \mu} = \bar{x} \left[\frac{1}{\delta - \mu} - \frac{1}{\frac{\sigma^2}{2}\xi^2 + \delta - \mu} \right] > 0.$$

  In other words, $\bar{x}$ is increasing in μ.

- For the comparative static w.r.t. δ, notice that:

$$\frac{\partial \xi}{\partial \delta} = \frac{\xi}{\frac{\sigma^2}{2}\xi^2 + \delta - \mu} > 0.$$

Thus ξ is increasing in δ. Then note that

$$\frac{d\bar{x}}{d\delta} = \frac{\partial \bar{x}}{\partial \xi}\frac{\partial \xi}{\partial \delta} + \frac{\partial \bar{x}}{\partial \delta}.$$

Since (i) ξ is increasing in δ (i.e. $\partial\xi/\partial\delta > 0$), (ii) $\bar{x}$ is decreasing in ξ (i.e. $\partial\bar{x}/\partial\xi < 0$), and (iii) keeping ξ constant, $\bar{x}$ is decreasing in δ (i.e. $\partial\bar{x}/\partial\delta < 0$), it means that $\bar{x}$ is decreasing in δ.

- For the comparative static w.r.t. α and θ, notice that ξ does not depend on those parameters, while $\bar{x}$ is decreasing in α and increasing in θ, delivering the result stated.
- The threshold $\bar{x}$ is trivially decreasing in κ, since ξ is independent of κ.
- For the comparative static w.r.t. m, note that:

$$\frac{\partial \xi}{\partial m} = \frac{\xi}{\sigma^2\xi - (m+\mu+\frac{\sigma^2}{2})} = \frac{\xi^2}{\frac{\sigma^2}{2}\xi^2 + \delta - \mu} > 0.$$

This means that

$$\frac{d\bar{x}}{dm} = \frac{\partial \bar{x}}{\partial \xi}\frac{\partial \xi}{\partial m} + \frac{\partial \bar{x}}{\partial m} = -\bar{x}\left[\frac{1}{\xi(\xi-1)}\frac{\partial \xi}{\partial m} + \frac{\delta-\kappa}{(\delta+m)(\kappa+m)}\right].$$

Note that ξ does not depend on κ, meaning that the term in brackets above is decreasing in κ. In other words, there exists $\bar{\kappa} > \delta$ (with $\bar{\kappa}$ potentially infinite) such that $\frac{d\bar{x}}{dm} < 0$ if and only if $\kappa < \bar{\kappa}$.

- Finally, consider the alternative state variable $z := (\kappa+m)x$, which allows us to abstract from the coupon rate κ. In that case, $\bar{z} = (\kappa+m)\bar{x}$ and we have

$$\frac{d\bar{z}}{dm} = \frac{\partial \bar{z}}{\partial \xi}\frac{\partial \xi}{\partial m} + \frac{\partial \bar{z}}{\partial m} = \bar{z}\left[\frac{1}{\delta+m} - \frac{1}{\xi(\xi-1)}\frac{\partial \xi}{\partial m}\right] = \bar{z}\left[\frac{1}{\delta+m} - \frac{1}{(\xi-1)\left(\sigma^2\xi - (m+\mu+\frac{\sigma^2}{2})\right)}\right].$$

Then note that:

$$(\xi-1)\left(\sigma^2\xi - (m+\mu+\frac{\sigma^2}{2})\right) - (\delta+m) = \frac{\sigma^2}{2}(\xi-1)^2 > 0.$$

Thus, $d\bar{z}/dm > 0$.

□

B.4.2 Speed of Reversion

As a reminder, the speed of reversion $\eta = \delta - r + (\xi-1)(m + \boldsymbol{\nu}\cdot\boldsymbol{\sigma})$. Since ξ is increasing in μ, it must be case that η is increasing in μ. Similarly, since ξ is increasing in δ, it must be case that η is increasing in δ. Note that α, θ and κ do not have any effect on ξ and thus on the speed of mean reversion η. We have established that ξ is increasing in m, which means that η must also be increasing in m. Finally, the comparative static w.r.t. σ cannot be signed. □

B.4.3 Attraction Point

As a reminder, the attraction point x_a is equal to

$$x_a = \bar{x}\left[\left(\frac{1-\alpha\theta}{1-\alpha\theta^{\xi}}\right)\left(\frac{\xi-1}{\delta-r}\left(m+\boldsymbol{\nu}\cdot\boldsymbol{\sigma}\right)+1\right)\right]^{\frac{1}{1-\xi}}.$$

The comparative statics w.r.t. r and ν are immediate, as neither $\bar{x}$ nor ξ depend on these parameters. For θ, note that $x_a \propto (1-\alpha)(1-\alpha\theta)^{\xi/(1-\xi)}(1-\alpha\theta^{\xi})^{-1/(1-\xi)} := K$, where $\propto$ is the "proportionality" sign. We then have

$$\frac{\partial K}{\partial \theta} = \frac{K\xi}{1-\xi}\left[\frac{\alpha\theta^{\xi-1}}{1-\alpha\theta^{\xi}} - \frac{\alpha}{1-\alpha\theta}\right].$$

The term in brackets is always negative and $\xi > 1$, implying that x_a is increasing in θ. □

B.4.4 Value Function

To perform those comparative statics, we leverage extensively Feynman-Kac and the integral representation of second order differential equations. Let us look at the comparative static w.r.t. κ for example. Remember that the value function v satisfies the following:

$$\begin{aligned}(\delta-\mu)v\left(x;\kappa\right) &= 1-(\kappa+m)x-(\mu+m)x\partial_x v\left(x;\kappa\right)+\frac{1}{2}|\sigma|^2x^2\partial_{xx}v\left(x;\kappa\right)\\ v\left(\bar{x};\kappa\right) &= \alpha v\left(\theta\bar{x};\kappa\right)\\ \partial_x v\left(\bar{x};\kappa\right) &= \alpha\theta\partial_x v\left(\theta\bar{x};\kappa\right)\end{aligned}$$

In the above, we have used a notation that emphasizes that the value function depends on the parameter κ. Differentiate the first two equations above w.r.t κ to obtain:

$$\begin{aligned}(\delta-\mu)\partial_\kappa v\left(x;\kappa\right) &= -x-(\mu+m)x\partial_{\kappa x}v\left(x;\kappa\right)+\frac{1}{2}|\sigma|^2x^2\partial_{\kappa xx}v\left(x;\kappa\right)\\ \frac{\partial\bar{x}}{\partial\kappa}\partial_x v\left(\bar{x};\kappa\right)+\partial_\kappa v\left(\bar{x};\kappa\right) &= \alpha\theta\frac{\partial\bar{x}}{\partial\kappa}\partial_x v\left(\theta\bar{x};\kappa\right)+\alpha\partial_\kappa v\left(\theta\bar{x};\kappa\right).\end{aligned}$$

Use the fact that $\partial_x v\left(\bar{x};\kappa\right) = \alpha\theta\partial_x v\left(\theta\bar{x};\kappa\right)$ to obtain the boundary condition $\partial_\kappa v\left(\bar{x};\kappa\right) = \alpha\partial_\kappa v\left(\theta\bar{x};\kappa\right)$. In other words, $\partial_\kappa v$ admits the following integral representation:

$$\partial_\kappa v(x) = \tilde{\mathbb{E}}_x^{nt}\left[\int_0^\infty \alpha^{N_{d,t}^{(\tau)}}e^{-(\delta-\mu)t}\left(-x_t\right)dt\right],$$

where $\tilde{\mathbb{E}}_x^{nt}$ is the expectation operator under the no-trade policy, in other words under which x_t evolves according to:

$$dx_t = -(m+\mu)x_t dt - x_t\boldsymbol{\sigma}\cdot d\tilde{\boldsymbol{B}}_t + (\theta-1)\,\bar{x}dN_{d,t}.$$

Thus, $\partial_\kappa v(x) < 0$. A similar method leads to the other comparative statics:

$$\begin{aligned}
\partial_{|\sigma|^2} v(x) &= \frac{1}{2}\tilde{\mathbb{E}}_x^{nt}\left[\int_0^\infty \alpha^{N_{d,t}^{(\tau)}} e^{-(\delta-\mu)t} x_t^2 v''(x_t)dt\right] > 0 \\
\partial_\delta v(x) &= -\tilde{\mathbb{E}}_x^{nt}\left[\int_0^\infty \alpha^{N_{d,t}^{(\tau)}} e^{-(\delta-\mu)t} v(x_t)dt\right] < 0 \\
\partial_\mu v(x) &= \tilde{\mathbb{E}}_x^{nt}\left[\int_0^\infty \alpha^{N_{d,t}^{(\tau)}} e^{-(\delta-\mu)t}\left(v(x_t) - x_t v'(x_t)\right)dt\right] > 0
\end{aligned}$$

For m we have

$$\partial_m v(x) = -\tilde{\mathbb{E}}_x^{nt}\left[\int_0^\infty \alpha^{N_{d,t}^{(\tau)}} e^{-(\delta-\mu)t} x_t\left(1 + v'(x_t)\right)dt\right],$$

and it suffices to study the sign of $1 + v'(x)$. Note that $v'(0) = -\frac{\kappa+m}{\delta+m}$, and since v is convex, we must have $v'(x) \geq -\frac{\kappa+m}{\delta+m}$ for all $x \in [0, \bar{x}]$. Thus, if $\kappa < \delta$, $v'(x) + 1 > 0$ for all $x \in [0, \bar{x}]$, meaning that $\partial_m v < 0$.

For the comparative statics w.r.t. α and θ, a slight modification of our proof is needed. For α for example, note that we have the following:

$$\begin{aligned}
(\delta-\mu)\partial_\alpha v(x;\alpha) &= -(\mu+m)x\partial_{\alpha x} v(x;\alpha) + \frac{1}{2}|\sigma|^2 x^2 \partial_{\alpha xx} v(x;\alpha) \\
\frac{\partial \bar{x}}{\partial \alpha}\partial_x v(\bar{x};\alpha) + \partial_\alpha v(\bar{x};\alpha) &= \alpha\theta\frac{\partial \bar{x}}{\partial \alpha}\partial_x v(\theta\bar{x};\alpha) + \alpha\partial_\alpha v(\theta\bar{x};\alpha) + v(\theta\bar{x};\alpha)
\end{aligned}$$

Use the smooth-pasting condition $\partial_x v(\bar{x};\alpha) = \alpha\theta\partial_x v(\theta\bar{x};\alpha)$ to obtain the boundary condition $\partial_\alpha v(\bar{x};\alpha) = \alpha\partial_\alpha v(\theta\bar{x};\alpha) + v(\theta\bar{x};\alpha)$. The differential equation satisfied by $\partial_\alpha v$ admits a "source" term equal to zero, while its terminal condition is the boundary condition for a function whose payoff is $v(\theta\bar{x})$ each time the barrier $\bar{x}$ is hit, with discount factor α. Thus, these equations taken together allow us to write the following integral representation for $\partial_\alpha v$:

$$\partial_\alpha v(x) = \tilde{\mathbb{E}}_x^{nt}\left[\sum_{k=1}^\infty e^{-(\delta-\mu)\tau_k}\alpha^k v(\theta\bar{x})\right] > 0$$

Similarly, one can show that

$$\partial_\theta v(x) = \tilde{\mathbb{E}}_x^{nt}\left[\sum_{k=1}^\infty e^{-(\delta-\mu)\tau_k}\alpha^{k+1}\bar{x}v'(\theta\bar{x})\right] < 0$$

Finally, consider the alternative state variable $z := (\kappa+m)x$, which allows us to abstract from the coupon rate κ. The value function $\tilde{v}(z) := v(z/(\kappa+m))$ then satisfies

$$\begin{aligned}
(\delta-\mu)\tilde{v}(z;m) &= 1 - z - (\mu+m)z\partial_z\tilde{v}(z;m) + \frac{1}{2}|\sigma|^2 z^2 \partial_{zz}\tilde{v}(z;m) \\
\tilde{v}(\bar{z};m) &= \alpha\tilde{v}(\theta\bar{z};m) \\
\partial_z\tilde{v}(\bar{z};m) &= \alpha\theta\partial_z v(\theta\bar{z};m).
\end{aligned}$$

Thus, we can express $\partial_m \tilde{v}$ as follows:

$$\partial_m \tilde{v}(z) = -\tilde{\mathbb{E}}_z^{nt}\left[\int_0^\infty \alpha^{N_{d,t}^{(\tau)}} e^{-(\delta-\mu)t} z_t \tilde{v}'(z_t)dt\right] > 0$$

$\square$

B.5 Uniqueness of the MPE under GBM

Our proof strategy largely follows DeMarzo and He (2021), with the important extension of possible buybacks along the equilibrium path in Section B.5.10.

B.5.1 Setup

We note Γ_t the cumulative bond issuance process, with $\Gamma_0 = 0$. Γ_t is a stochastic process that must be adapted to the filtration $\mathcal{F}_t$, σ-algebra generated by $\{B_u, 0 \le u \le t\}$. The face value $F_t^{(\Gamma)}$ resulting from such bond issuance process is:

$$F_t^{(\Gamma)} = e^{-mt}F_0 + \int_0^t e^{m(u-t)} d\Gamma_u.$$

This implies that $dF_t = d\Gamma_t - mF_{t-}dt$. For any policy (Γ, τ) followed by the government, the debt price is defined via:

$$D(Y,F) = \mathbb{E}_{Y,F}^{\mathbb{Q}}\left[\int_0^\tau e^{-(r+m)t}(\kappa+m)dt + e^{-(r+m)\tau}\underline{D}(Y_\tau, F_\tau)\right],$$

where $\underline{D}$ is the debt recovery value in default, with $\underline{D}(Y,F) := \underline{d}(F/Y)$. Under the probability measure $\Pr^{\mathbb{Q}}$, $B_t^{\mathbb{Q}} := B_t - \nu t$ is a Brownian motion. The government solves the following problem:

$$V(Y,F) = \sup_{(\Gamma,\tau)\in\mathcal{G}\times\mathcal{T}} \mathbb{E}_{Y,F}\left[\int_0^\tau e^{-\delta t}\left(\left(Y_t - (\kappa+m)F_t^{(\Gamma)}\right)dt + D\left(Y_t, F_t^{(\Gamma)}\right)d\Gamma_t\right) + e^{-\delta\tau}\underline{V}\left(Y_\tau, F_\tau\right)\right].$$

$\underline{V}(Y,F) := Y\underline{v}(F/Y)$ is the default value for the government, with $\underline{d}(x) + \underline{v}'(x) = 0$ by assumption.

B.5.2 Scalability

Let x_t be the debt-to-income ratio: $x_t := F_t/Y_t$. We restrict ourselves to the study of Markov equilibria in the state variable x_t. Note that $x_t^{(\gamma)}$ is equal to:

$$x_t^{(\gamma)} = x_0 + \gamma_t - \int_0^t \left(m + \mu - \sigma^2\right) x_u^{(\gamma)} ds - \int_0^t \sigma x_u^{(\gamma)} dB_u.$$

In the above, we have introduced the scaled issuance policy γ_t, defined as follows:

$$\gamma_t := \int_0^t \frac{d\Gamma_u}{Y_u}.$$

If the debt price is homogenous of degree zero in (Y, F), then it is optimal for the government to follow a financing and default policies that are homogeneous of degree 1, because:

$$\begin{aligned}
V(Y,F) &= \sup_{(\Gamma,\tau)\in\mathcal{G}\times\mathcal{T}} \mathbb{E}_{Y,F}\left[\int_0^\tau e^{-\delta t}\left(\left(Y_t-(\kappa+m)F_t^{(\Gamma)}\right)dt + D\left(Y_t,F_t^{(\Gamma)}\right)d\Gamma_t\right) + e^{-\delta\tau}\underline{V}\left(Y_\tau,F_\tau\right)\right] \\
&= Y \sup_{(\gamma,\tau)\in\mathcal{G}\times\mathcal{T}} \mathbb{E}_x\left[\int_0^\tau \frac{M_t}{M_0}e^{-(\delta-\mu)t}\left(\left(1-(\kappa+m)x_t^{(\gamma)}\right)dt + d\left(x_t^{(\gamma)}\right)d\gamma_t\right) + \frac{M_\tau}{M_0}e^{-(\delta-\mu)\tau}\underline{v}\left(x_\tau^{(\gamma)}\right)\right] \\
&= Y \sup_{(\gamma,\tau)\in\mathcal{G}\times\mathcal{T}} \tilde{\mathbb{E}}_x\left[\int_0^\tau e^{-(\delta-\mu)t}\left(\left(1-(\kappa+m)x_t^{(\gamma)}\right)dt + d\left(x_t^{(\gamma)}\right)d\gamma_t\right) + e^{-(\delta-\mu)\tau}\underline{v}\left(x_\tau^{(\gamma)}\right)\right],
\end{aligned}$$

where the second equality follows from $\frac{Y_t}{Y_0} = \frac{M_t}{M_0}e^{\mu t}$, where $M_t = \exp\left(-\frac{\sigma^2}{2}t + \sigma B_t\right)$ is a strictly positive martingale that induces the usual change in measure. This allows us to define the scaled value function:

$$v(x) := \sup_{(\gamma,\tau)\in\mathcal{G}\times\mathcal{T}} \tilde{\mathbb{E}}_x\left[\int_0^\tau e^{-(\delta-\mu)t}\left(\left(1-(\kappa+m)x_t^{(\gamma)}\right)dt + d\left(x_t^{(\gamma)}\right)d\gamma_t\right) + e^{-(\delta-\mu)\tau}\underline{v}\left(x_\tau^{(\gamma)}\right)\right].$$

Under the change of measure $\tilde{\Pr}$, $\tilde{B}_t := B_t - \sigma t$ is a Brownian motion, and x_t follows

$$x_t^{(\gamma)} = x_0 + \gamma_t - \int_0^t (m+\mu)\, x_u^{(\gamma)} du - \int_0^t \sigma x_u^{(\gamma)} d\tilde{B}_u.$$

Similarly, if the default and issuance policies are homogeneous of degree 1, the debt price is homogeneous of degree zero. The debt price can then be written:

$$d(x) = \mathbb{E}_x^{\mathbb{Q}}\left[\int_0^\tau e^{-(r+m)t}(\kappa+m)dt + e^{-(r+m)\tau}\underline{d}\left(x_\tau^{(\gamma)}\right)\right].$$

B.5.3 Inequalities satisfied by d and v

We then note that we must have:

$$0 \le d(x) \le \frac{\kappa+m}{r+m} \qquad \forall x. \tag{21}$$

In any equilibrium, we must also have:

$$v(x) \ge \underline{v}(x) \tag{22}$$

$$v(x) \ge \max_{x'} v(x') + \left(x' - x\right) d(x'). \tag{23}$$

The first inequality is simply saying that the government always has the option to default. The second inequality says that the government can always jump to a debt-to-income ratio x'.

B.5.4 Convexity of v

The inequality (23) leads us to conclude that v must be convex. Indeed, take two arbitrary debt-to-income ratios x_1, x_2, and $\lambda \in [0,1]$, with $x_\lambda := \lambda x_1 + (1-\lambda)x_2$. Consider feasible policies that make the government jump from x_1 to x_λ, or from x_2 to x_λ. Then we have:

$$v(x_1) \geq v(x_\lambda) + (x_\lambda - x_1)\, d\,(x_\lambda)$$
$$v(x_2) \geq v(x_\lambda) + (x_\lambda - x_2)\, d\,(x_\lambda)$$

Take a weighted average of these two inequalities to obtain:

$$\lambda v(x_1) + (1-\lambda)v(x_2) \geq v(x_\lambda).$$

B.5.5 Monotonicity of v and Cutoff Policy for Default

Take $x' > x$. Using inequality (23) we know v is non-increasing:

$$v(x) \geq v(x') + (x' - x)d(x') \geq v(x'). \tag{24}$$

The second inequality follows from $x' > x$ and the non-negativity of debt price $d(x')$ in (21). Thus v is non-increasing in x. Finally, note that on the subset $\{x > 0 : d(x) > \underline{d}(x)\}$, the value function v is strictly decreasing. As a result, in any equilibrium there exists $\bar{x} > 0$ such that $d(x) = \underline{d}(x)$ and the government defaults at x if and only if $x \geq \bar{x}$. It is then straightforward to show that $\bar{x} < +\infty$; indeed, if it was not the case, debt would be risk-free, its price would be equal to $(\kappa + m)/(r+m) > 0$, and we could then use inequality (24) to show that if x is large enough, $v(x) < \underline{v}(x)$, giving an incentive for the government to default for some finite x, leading to a contradiction. Thus, the default boundary $\bar{x}$ must be finite.

B.5.6 No Convex Kinks

Define the left and right derivatives of v, noted v'_- and v'_+ respectively, as follows:

$$v'_+(x) := \lim_{h \downarrow 0} \frac{v(x+h) - v(x)}{h} \qquad\qquad v'_-(x) := \lim_{h \uparrow 0} \frac{v(x+h) - v(x)}{h}$$

We know that $v'_+(x_1) \leq v'_-(x_2) \leq v'_+(x_2)$ for all $x_1 < x_2$, thanks to the convexity of v shown in Section B.5.4. We want to show that v cannot have a convex kink; this implies that v is differentiable everywhere. Intuitively, suppose that v had a convex kink at x_0, then $v'_+(x_0) > v'_-(x_0)$. In such case, consider a financing policy under which the government does not issue any bonds in the neighborhood of x_0. We introduce the second derivative measure $\nu(\cdot)$, defined for $x_1 < x_2$ via:

$$\nu\left([x_1, x_2)\right) := v'_-(x_2) - v'_-(x_1).$$

If v'' exists at x, then $\nu(dx) = v''(x)dx$. Finally, the local time $L_t(x)$ is defined via:

$$L_t(x) := \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{\{|x_u - x| < \epsilon\}} du.$$

We have established that v is convex, and thus $\mathcal{C}^1$ except maybe at a countable number of points. We prove that v cannot feature convex kinks in two steps. First we focus on isolated kinks, establishing a contradiction. We then focus on kinks that are not isolated.

Let us first consider isolated kinks. Assume that v is not differentiable at x_0, and take ϵ small enough that v is $\mathcal{C}^1$ everwhere on $[x_0 - \epsilon, x_0 + \epsilon]$, except at x_0. Consider once again the strategy γ_0, of not issuing any bonds until the first time x_t hits either $x_0 - \epsilon$ or $x_0 + \epsilon$. We then have the generalized Ito-Tanaka formula for convex functions (see Karatzas and Shreve, Theorem 6.22 and its generalization by Elworthy, Truman and Zhao (2007)):

$$
\begin{aligned}
e^{-(\delta-\mu)\tau_\epsilon} v(x^{(\gamma_0)}_{\tau_\epsilon}) = v(x_0) - \int_0^{\tau_\epsilon} e^{-(\delta-\mu)u} \left[(m+\mu) x^{(\gamma_0)}_u v'_-(x^{(\gamma_0)}_u) + (\delta-\mu) v(x^{(\gamma_0)}_u) \right] du \\
+ \int_{x_0-\epsilon}^{x_0+\epsilon} \int_0^{\tau_\epsilon} e^{-(\delta-\mu)u} \frac{\sigma^2 y^2}{2} L_u\,(y)\, \nu(dy) du \\
- \sigma \int_0^{\tau_\epsilon} e^{-(\delta-\mu)u} x_u v'_-(x^{(\gamma_0)}_u) d\tilde{B}_u
\end{aligned} \tag{25}
$$

Note that if v was twice differentiable on $[x_0 - \epsilon, x_0 + \epsilon]$, the formula above would become:

$$
\begin{aligned}
e^{-(\delta-\mu)\tau_\epsilon} v(x^{(\gamma_0)}_{\tau_\epsilon}) = v(x_0) + \int_0^{\tau_\epsilon} e^{-(\delta-\mu)u} \left(\hat{\mathcal{A}} - (\delta-\mu)\right) v(x^{(\gamma_0)}_u) du \\
+ \frac{\sigma^2 x_0^2}{2} \left(v'_+(x_0) - v'_-(x_0) \right) \int_0^{\tau_\epsilon} dL_t(x_0) - \sigma \int_0^{\tau_\epsilon} e^{-(\delta-\mu)u} x^{(\gamma_0)}_u v'(x^{(\gamma_0)}_u) d\tilde{B}_u.
\end{aligned}
$$

Take expectations of (25) to obtain:

$$
\begin{aligned}
\mathbb{E}\left[e^{-(\delta-\mu)\tau_\epsilon} v(x^{(\gamma_0)}_{\tau_\epsilon}) \right] = v(x_0) + \mathbb{E}\Bigg[\int_{x_0-\epsilon}^{x_0+\epsilon} \int_0^{\tau_\epsilon} e^{-(\delta-\mu)u} \frac{\sigma^2 y^2}{2} L_u\,(y)\, \nu(dy) du \\
- \int_0^{\tau_\epsilon} e^{-(\delta-\mu)u} \left[(m+\mu) x^{(\gamma_0)}_u v'_-(x^{(\gamma_0)}_u) + (\delta-\mu) v(x^{(\gamma_0)}_u) \right] du \Bigg]
\end{aligned}
$$

By taking ϵ small enough, we can make the left-handside of the equation above strictly greater than the right-handside, as the first term in the above expectation ends up dominating the second term. This would lead to the desired contradiction.

If the kink at x_0 is not isolated, we can simply consider the convex hull $\tilde{v}$ of v that satisfies $v(x_0) = \tilde{v}(x_0)$ and that is piece-wise linear. One can apply the above reasoning to $\tilde{v}$ rather than v, and notice that the convex hull $\tilde{v}$ is below the original function v at all points in the neighborhood of x_0 (except at x_0, where the functions are equal), in order to obtain the desired conclusion.

B.5.7 No-trade value as a lower bound on v

Under the no-trade policy, notice that x_t satisfies:

$$
x_t = x_0 \exp\left(- \left(m + \mu + \frac{\sigma^2}{2} \right) t - \sigma \tilde{B}_t \right)
$$

Thus, if we note v_0 the government no-trade value, we have:

$$
\begin{aligned}
v_0(x) := \sup_{\tau \in \mathcal{T}} \tilde{\mathbb{E}} \left[\int_0^\tau e^{-(\delta-\mu)t} \left(1 - (\kappa+m) x e^{-\left(m+\mu+\frac{\sigma^2}{2}\right)t - \sigma \tilde{B}_t} \right) dt + e^{-(\delta-\mu)\tau} \underline{v}\left(x e^{-\left(m+\mu+\frac{\sigma^2}{2}\right)\tau - \sigma \tilde{B}_\tau} \right) \right] \quad (26) \\
= \varphi(x) + \left(\frac{x}{\bar{x}_0} \right)^{\xi} \left[\underline{v}(\bar{x}_0) - \varphi\left(\bar{x}_0\right) \right] \quad (27)
\end{aligned}
$$

$\bar{x}_0$ is the optimal debt-to-income default boundary under the assumption that no bond is ever issued, satisfying $v_0'(\bar{x}_0) := \underline{v}'(\bar{x}_0)$. $\varphi(x)$ is the sovereign value under no-trade and no-default, which satisfies

$$\varphi(x) := \frac{1}{\delta - \mu} - \left(\frac{\kappa + m}{\delta + m}\right) x$$

In any equilibrium, for any x, we have the following inequality:

$$v(x) \geq v_0(x).$$

In other words, the government can simply "do nothing," and not issue any bond ever again. Finally, this must also mean that $\bar{x} \geq \bar{x}_0$.

B.5.8 Equilibrium debt pricing $d(x) + v'(x) = 0$

For any $x, x_0 < \bar{x}$, from equation (23) we must have

$$v(x) - v(x_0) \geq (-d(x_0))\,(x - x_0)$$

In other words, $-d(x_0)$ belongs to the subdifferential of v at x_0. Since v is differentiable everywhere, it must be the case that the subdifferential of v is made up of only one point, i.e., $d(x) = -v'(x)$.

B.5.9 Monotonicity and Differentiability of d

From Rademacher theorem, the gradient $-d$ of v must be continuous on the set of points where v is differentiable. Since v is differentiable everywhere, it follows that d is continuous in x. The convexity of v then implies the monotonicity of v', meaning that d must be monotone weakly decreasing in x. From Lebesgue's theorem for the differentiability of monotone functions, d must be differentiable almost everywhere (which also means that v' is differentiable almost everywhere).

B.5.10 Issuance and Buy-Backs

As an important intermediate step for establishing uniqueness, we rule out certain types of financing behavior for the government, depending on the magnitude of bond risk premia. More specifically, we show that there exist three regions of the state space that form a partition; and in each such region, it is either (a) never optimal to buy back debt (when the risk-premium earned by international investors is sufficiently low), (b) never optimal to issue debt (when the risk-premium earned by international investors is sufficiently high); or, (c) never optimal to trade (when debt risk-premia are exactly equal to the wedge $\delta - r$). This means that the cumulative debt issuance process, in each region of the partition, is monotone and the Lebesgue decomposition theorem applies region by region, ruling out Brownian shocks to the cumulative issuance process Γ_t.

Note $D_t^{(\Gamma)} := D\left(Y_t, F_t^{(\Gamma)}\right)$. Recall that, for any issuance policy Γ, the risk-premium earned by international creditors is:

$$\pi\left(Y_t, F_t^{(\Gamma)}\right) dt := \mathbb{E}_t\left[\frac{dD_t^{(\Gamma)} + (\kappa + m)dt}{D_t^{(\Gamma)}}\right] - (r + m)dt. \tag{28}$$

Since d is differentiable almost everywhere (see Section B.5.9), on such points of differentiability we know that $\pi(Y,F) = \pi(x) = -(\boldsymbol{\nu}(s)\cdot\boldsymbol{\sigma})xd'(x)/d(x)$. Consider the sets $\mathcal{O}_+, \mathcal{O}_-, \mathcal{O}_0$, defined as follows:

$$\mathcal{O}_+ := \{(Y,F) : \delta - r > \pi(Y,F)\}$$
$$\mathcal{O}_- := \{(Y,F) : \delta - r < \pi(Y,F)\}$$
$$\mathcal{O}_0 := \{(Y,F) : \delta - r = \pi(Y,F)\}.$$

Note that $(\mathcal{O}_+, \mathcal{O}_-, \mathcal{O}_0)$ form a partition of the state space.

This part of our proof has four steps: (i) in $\mathcal{O}_+$, the government always wants to issue debt, whereas in $\mathcal{O}_-$ the government always wants to buy back debt; (ii) in $\mathcal{O}_0$, it is strictly optimal for the government to stay put; (iii) we apply the Lebesgue decomposition theorem to write down the process of x as the difference between two monotone processes; (iv) we rule out the lump-sum debt issuance at the default. For steps (i)-(ii), we assume that the government does not issue a positive measure of debt and immediately defaults afterwards; we verify that this type of behavior is suboptimal in step (iv).

(i) No buy-back in $\mathcal{O}_+$ and no issuance in $\mathcal{O}_-$. We use a proof by contradiction. Suppose for an instant that the government issuance strategy Γ features a debt-buyback in $\mathcal{O}_+$, or a debt issuance in $\mathcal{O}_-$ (and strictly so in one of the regions). To arrive at a contradiction, we will construct an alternative policy $\hat{\Gamma}$ that strictly improves upon Γ.

Let τ be the default time under the *original* policy. For now, we assume that the government does not make a lump-sum debt issuance at such default time τ; in step (iv) below, we show that such "jump-to-default" debt issuances would be suboptimal. Define the following stopping times:

$$\tau_t^{\mathcal{O}_+} := \inf_{s\le t}\{s : (Y_u, F_u) \in \mathcal{O}_+, u \in (s,t)\}$$
$$\tau_t^{\mathcal{O}_0} := \inf_{s\le t}\{s : (Y_u, F_u) \in \mathcal{O}_0, u \in (s,t)\}$$
$$\tau_t^{\mathcal{O}_-} := \inf_{s\le t}\{s : (Y_u, F_u) \in \mathcal{O}_-, u \in (s,t)\}$$

Essentially, these stopping times are the most recent entry times into the three different regions. Consider then the alternative financing strategy $\hat{\Gamma}$, built as follows:

$$F_t^{\hat{\Gamma}} = \begin{cases} \max\left\{e^{-m(t-\tau_t^{\mathcal{O}_+})}F^{\hat{\Gamma}}_{\tau_t^{\mathcal{O}_+}}, \sup_{\tau_t^{\mathcal{O}_+}\le s\le t}\{e^{-m(t-s)}F_s^{(\Gamma)}\}\right\} & \text{when } (Y_t,F_t)\in\mathcal{O}_+ \\ e^{-m(t-\tau_t^{\mathcal{O}_0})}F^{\hat{\Gamma}}_{\tau_t^{\mathcal{O}_0}} & \text{when } (Y_t,F_t)\in\mathcal{O}_0 \\ \min\left\{e^{-m(t-\tau_t^{\mathcal{O}_-})}F^{\hat{\Gamma}}_{\tau_t^{\mathcal{O}_-}}, \inf_{\tau_t^{\mathcal{O}_-}\le s\le t}\{e^{-m(t-s)}F_s^{(\Gamma)}\}\right\} & \text{when } (Y_t,F_t)\in\mathcal{O}_- \end{cases} \tag{29}$$

$$d\hat{\Gamma}_\tau = \max\{F^{\hat{\Gamma}}_{\tau_-} - F^{\Gamma}_{\tau}, 0\} \tag{30}$$

The alternative strategy postpones any buybacks when $(Y_t, F_t) \in \mathcal{O}_+$, stays put when (Y_t, F_t) enters $\mathcal{O}_0$, and postpones issuances when $(Y_t, F_t) \in \mathcal{O}_-$; and trading occurs (i.e., issuances in $\mathcal{O}_+$ and/or buybacks in $\mathcal{O}_-$) only when the alternative policy catches up with the original policy. Finally, at the default time τ, whenever the alternative policy induces a weakly lower debt balance than that under original policy at such terminal time, under the alternative policy $\hat{\Gamma}$ a lumpy amount of debt is issued to insure that the terminal face value under the original and alternative policies are identical. This immediately implies that $F_\tau^{\hat{\Gamma}} \geq F_\tau^{\Gamma}$.

There are two key properties of this alternative strategy. First, we must have

$$D\left(Y_t, F_t^{(\hat{\Gamma})}\right) d\hat{\Gamma}_t = D\left(Y_t, F_t^{(\Gamma)}\right) d\hat{\Gamma}_t. \tag{31}$$

In other words, the debt proceeds obtained between t and $t + dt$ under the alternative strategy $\hat{\Gamma}$ (the left hand-side of equation (31)) must be equal to the proceeds that would have hypothetically been obtained under $\hat{\Gamma}$ if prices were wrongly computed assuming the debt policy Γ (the right hand-side of equation (31)). This is because either $d\hat{\Gamma}_t = 0$ so equation (31) holds automatically, or $F_t^{(\hat{\Gamma})} = F_t^{(\Gamma)}$ whenever $d\hat{\Gamma}_t \neq 0$ (for a similar argument, see DeMarzo and He (2021)).

The second important property of this alternative strategy is that

$$\left(\delta - r - \pi\left(Y_t, F_t^{(\Gamma)}\right)\right)\left[F_t^{(\hat{\Gamma})} - F_t^{(\Gamma)}\right] D\left(Y_t, F_t^{(\Gamma)}\right) \geq 0. \tag{32}$$

More specifically, we have (positive or negative are weakly)

$$\Delta F_t := F_t^{(\hat{\Gamma})} - F_t^{(\Gamma)} \in \begin{cases} \mathbb{R}^+ & \text{when } (Y_t, F_t) \in \mathcal{O}_+ \text{ i.e., } \delta - r - \pi\left(Y_t, F_t^{(\Gamma)}\right) > 0; \\ \mathbb{R} & \text{when } (Y_t, F_t) \in \mathcal{O}_+ \text{ i.e., } \delta - r - \pi\left(Y_t, F_t^{(\Gamma)}\right) = 0; \\ \mathbb{R}^- & \text{when } (Y_t, F_t) \in \mathcal{O}_- \text{ i.e., } \delta - r - \pi\left(Y_t, F_t^{(\Gamma)}\right) < 0. \end{cases}$$

In other words, in the low (resp. high) risk-premium region, our alternative policy $\hat{\Gamma}$ must have higher (resp. lower) debt than our original policy Γ. Let us now show that this alternative strategy $\hat{\Gamma}$ is a strict improvement upon Γ. Consider the difference in payoffs $\Delta V_{[0;\tau]}$ between using strategy Γ and using strategy $\hat{\Gamma}$, over the time interval $[0, \tau)$. Below we show that this payoff difference $\Delta V_{[0;\tau]} > 0$, providing a contradiction:

$$\begin{aligned} \Delta V_{[0;\tau]} :&= \mathbb{E}_{Y,F}\left[\int_0^\tau e^{-\delta t}\left[(\kappa + m)\left(F_t^{(\Gamma)} - F_t^{(\hat{\Gamma})}\right) dt + D\left(Y_t, F_t^{(\hat{\Gamma})}\right) d\hat{\Gamma}_t - D\left(Y_t, F_t^{(\Gamma)}\right) d\Gamma_t\right]\right] \\ &= \mathbb{E}_{Y,F}\left[\int_0^\tau e^{-\delta t}\left[-(\kappa + m)\Delta F_t dt + D\left(Y_t, F_t^{(\Gamma)}\right) d\Delta\Gamma_t\right]\right] \\ &= \mathbb{E}_{Y,F}\left[\int_0^\tau e^{-\delta t}\left[-(\kappa + m)\Delta F_t dt + D\left(Y_t, F_t^{(\Gamma)}\right)(d\Delta F_t + m\Delta F_t dt)\right]\right]. \end{aligned} \tag{33}$$

In order to go from the first to the second line above, we have used our observation (31). We can then integrate by parts and use the debt pricing equation (28) to obtain

$$\Delta V_{[0;\tau]} = \mathbb{E}_{Y,F}\left[\int_0^\tau e^{-\delta t}\left(\delta - r - \pi\left(Y_t, F_t^{(\Gamma)}\right)\right)\Delta F_t D_t^{(\Gamma)} dt + e^{-\delta\tau}\Delta F_\tau D_\tau^{(\Gamma)} - \Delta F_0 D_0^{(\Gamma)}\right].$$

Under our alternative strategy $\hat{\Gamma}$ we must have inequality (32), strict on a positive measure of time by assumption; moreover, $\Delta F_0 = 0$, and finally $\Delta F_\tau \geq 0$ (recall equation (30)) and $D_\tau^{(\Gamma)} \geq 0$. Thus, $\Delta V_{[0;\tau]} > 0$, which means we constructed a profitable deviation – a contradiction.

(ii) No trading in $\mathcal{O}_0$. We then focus on the region of the state space $\mathcal{O}_0$, and show that in such region, it is never optimal for the government to issue or buy-back debt. Suppose that in equilibrium the government spent a strictly positive measure of time inside the set $\mathcal{O}_0$ where $\delta = r + \pi\left(Y_t, F_t^{(\Gamma)}\right)$. Since $\pi\left(Y_t, F_t^{(\Gamma)}\right) = \delta - r > 0$, in that region the debt price $D(Y, F)$ is strictly decreasing in the debt balance F. It is then immediate that the equilibrium debt issuance policy has continuous sample path; otherwise, the government is strictly worse off by trading an amount of debt with a discrete size $|d\Gamma| > \epsilon > 0$:

$$\underbrace{V(Y,F)}_{\text{equilibrium value}} \overset{\text{strict convexity}}{>} V(Y, F + d\Gamma) + d\Gamma \cdot V_F(Y, F + d\Gamma)$$
$$= \underbrace{V(Y, F + d\Gamma) + d\Gamma \cdot D(Y, F + d\Gamma)}_{\text{value from trading } d\Gamma}.$$

We claim that there must be no trading over $\mathcal{O}_0$. Suppose otherwise; without loss of generality, consider $s \in [t_0, t_1]$ so that $(Y_s, F_s) \in \mathcal{O}_0$ for all $s \in [t_0, t_1]$, and

$$d\Gamma_s > 0 \text{ for some strictly positive measure of time.} \tag{34}$$

Consider the alternative policy indexed by a constant $k \in (0, 1)$, which is exactly the same outside the time interval $[t_0, t_1]$, but inside the interval,

$$d\hat{\Gamma}_s = \begin{cases} kd\Gamma_s & \text{when } d\Gamma_s > 0; \\ d\Gamma_s \cdot 1_{\Delta F_s \geq 0} & \text{when } d\Gamma_s \leq 0. \end{cases} \tag{35}$$

In other words, when debt is issued under the original policy ($d\Gamma_s > 0$), a strictly lower amount of debt is issued under the alternative policy – this occurs during a strictly positive measure of time given (34). When debt is repurchased under the original policy ($d\Gamma_s \leq 0$), the government either (i) stays put if the debt balance $F_s^{(\hat{\Gamma})} < F_s^{(\Gamma)}$ is strictly lower under the alternative policy, or (ii) buys back a minimum amount of debt to make the debt balances equal under the original and alternative policies, so that $\Delta F_s = 0$ and $d\hat{\Gamma}_s = d\Gamma_s \leq 0$. (Note, since Γ_s has a continuous sample path, $d\hat{\Gamma}_s < 0$ could occur only when $\Delta F_s = 0$.)

It is easy to show that this alternative policy induces a lower debt balance for $t \in [t_0, t_1]$:

$$\Delta F_s = F_s^{(\hat{\Gamma})} - F_s^{(\Gamma)} \leq 0 \text{ and strictly so for some } s. \tag{36}$$

Importantly, we have the following two key properties for the alternative policy. First, because the alternative policy buys back debt only when the resulting debt balance is exactly the same as the one in the original policy, the same logic as in equation (31) implies that

$$D\left(Y_s, F_s^{(\hat{\Gamma})}\right) d\hat{\Gamma}_s = D\left(Y_s, F_s^{(\Gamma)}\right) d\hat{\Gamma}_s \text{ when } d\hat{\Gamma}_s \leq 0. \tag{37}$$

Second, because the debt price is strictly decreasing in F in the region $\mathcal{O}_0$, we have

$$D\left(Y_s, F_s^{(\hat{\Gamma})}\right) d\hat{\Gamma}_s \geq D\left(Y_s, F_s^{(\Gamma)}\right) d\hat{\Gamma}_s \text{ when } d\hat{\Gamma}_s > 0 \text{ and strictly so for some } s. \tag{38}$$

Moreover, this inequality holds on a strictly positive measure of time. The weak inequality in (38) is obvious since $d\hat{\Gamma}_s > 0$ and $\Delta F_s \leq 0$ as shown in (36). We establish that this inequality holds over a strictly positive measure of time by contradiction. Define the set $S \subset [t_0, t_1]$ so that $d\Gamma_s > 0$ for all $s \in S$, which has a strictly positive measure by assumption. Suppose, counterfactually, that $F_s^{(\hat{\Gamma})} - F_s^{(\Gamma)} = 0$ always for $s \in S$. Then on the complement set of $S^c \equiv [t_0, t_1] \setminus S$, $d\Gamma_s \leq 0$, and from the construction of the alternative policy in (35) we have $0 \leq d\Gamma_s \leq d\hat{\Gamma}_s$. Integrating the debt issuances over S^c, we conclude that $F_s^{(\hat{\Gamma})} \geq F_s^{(\Gamma)}$ always—but this contradicts with (36). This proves the strict inequality in (38).

We can then evaluate the gains from the alternative policy relative to the original one:

$$\begin{aligned}
\Delta V_{[t_0,t_1]} &= \mathbb{E}_{t_0}\left[\int_{t_0}^{t_1} e^{-\delta s}\left[(\kappa+m)\left(F_s^{(\Gamma)} - F_s^{(\hat{\Gamma})}\right)dt + D\left(Y_s, F_s^{(\hat{\Gamma})}\right)d\hat{\Gamma}_s - D\left(Y_s, F_s^{(\Gamma)}\right)d\Gamma_s\right]\right] \\
&> \mathbb{E}_{t_0}\left[\int_{t_0}^{t_1} e^{-\delta s}\left[(\kappa+m)\left(F_s^{(\Gamma)} - F_s^{(\hat{\Gamma})}\right)dt + D\left(Y_s, F_s^{(\Gamma)}\right)d\hat{\Gamma}_s - D\left(Y_s, F_s^{(\Gamma)}\right)d\Gamma_s\right]\right] \\
&= \mathbb{E}_{t_0}\left[\int_{t_0}^{t_1} e^{-\delta s}\left\{\begin{array}{c} \underbrace{\left(\delta - r - \pi\left(Y_s, F_s^{(\Gamma)}\right)\right)}_{0}\Delta F_s D_s^{(\Gamma)} ds \\ +e^{\delta t_1}\underbrace{\Delta F_{t_1} D\left(Y_{t_1}, F_{t_1}^{(\Gamma)}\right)}_{0} - \underbrace{\Delta F_{t_0} D\left(Y_{t_0}, F_{t_0}^{(\Gamma)}\right)}_{0} \end{array}\right\} ds\right] = 0.
\end{aligned}$$

This is a contradiction. A similar reasoning holds for buy-backs. Thus, we have proven by contradiction that on the set $\mathcal{O}_0$, the government's optimal policy is "stay put."

(iii) Lebesgue decomposition theorem and evolution of x. We can then conclude, in the general case $\nu \neq 0$, that the optimal policy Γ can be written as the difference between two monotone increasing processes Γ^+ and Γ^-. The increasing process Γ^+ represents bond issuances whenever $(Y, F) \in \mathcal{O}_+$, while the increasing process Γ^- represents bond repurchases whenever $(Y, F) \in \mathcal{O}_-$. Thus, $\Gamma = \Gamma^+ - \Gamma^-$. We then use Lebesgue decomposition theorem for monotone functions to express Γ^+ and Γ^- as follows:

$$\begin{aligned}
\Gamma_t^+ &= \Gamma_t^{ac+} + \Gamma_t^{pp+} + \Gamma_t^{sing+} \\
\Gamma_t^- &= \Gamma_t^{ac-} + \Gamma_t^{pp-} + \Gamma_t^{sing-}.
\end{aligned}$$

Γ^{pp+} (resp. Γ^{pp-}) is the pure point part of Γ^+ (resp. Γ^-), wile Γ^{sing+} (resp. Γ^{sing-}) is the singular continuous part of Γ^+ (resp. Γ^-). Finally Γ_t^{ac+} and Γ_t^{ac-} are two absolutely continuous and increasing processes. Define γ_u^{pp+} as scaled version of Γ_u^{pp+}, and accordingly for γ_u^{pp-}, γ_u^{sing+}, and γ_u^{sing-}. The debt-to-income process x_t must then satisfy:

$$\begin{aligned}
x_t^{(\gamma)} = x_0 + \int_0^t \left(\gamma_u - \left(m + \mu - \sigma^2\right) x_u^{(\gamma)}\right) du - \int_0^t \sigma x_u^{(\gamma)} dB_u \\
+ \int_0^t \left(d\gamma_u^{pp+} - d\gamma_u^{pp-}\right) + \int_0^t \left(d\gamma_u^{sing+} - d\gamma_u^{sing-}\right).
\end{aligned} \tag{39}$$

(iv) Strict suboptimality of "jump to default" strategies. Note that the process for x in (39) allows for the possibility of jumps to default, i.e., the state variable x jumps from, say, $x_1 < \bar{x}$ to $\bar{x}$ at the default time τ. However, because the equilibrium debt price must be $d(\bar{x}) = \underline{d}(\bar{x}) = -\underline{v}'(\bar{x})$, this implies that at x_1 we have

$$v(x_1) = \underline{v}(\bar{x}) + d(\bar{x})(\bar{x} - x_1) = \underline{v}(\bar{x}) + \underline{v}'(\bar{x})(x_1 - \bar{x}) < v_0(x_1),$$

due to the strict convexity of $v_0(\cdot)$ established in equation (27). But the government can achieve $v_0(x_1)$ by staying put, a contradiction.

B.5.11 Hamilton Jacobi Bellman Equation in Smooth Region

The preceding results show that whenever the state variable x is not at a point of singular issuance or buy-back (i.e. at a point where $d\gamma_t^{sing+} = d\gamma_t^{sing-} = 0$) or at a point where a jump is optimal (i.e. at a point where $d\gamma_t^{pp+} = d\gamma_t^{pp-} = 0$), the following Hamilton Jacobi Bellman must hold:

$$1 - (\kappa + m)x + (\mathcal{A} - (\delta - \mu))\, v(x) = 0, \tag{40}$$

where the operator $\mathcal{A}$ is defined for any $f \in \mathcal{C}^2$ via:

$$\mathcal{A}f(x) := -\,(m + \mu)\, x f'(x) + \frac{\sigma^2 x^2}{2} f''(x).$$

In the interior of any interval where equation (40) holds, v must be $\mathcal{C}^\infty$, since one can express $v^{(k+2)}$ as a function of $v^{(k+1)}$ and $v^{(k)}$ for any $k \geq 0$. When differentiating equation (40) w.r.t. x, we obtain (equivalently, using $v'(x) + d(x) = 0$)

$$(\delta + m)v'(x) = -(\kappa + m) - \left(m + \mu - \sigma^2\right) x v''(x) + \frac{\sigma^2 x^2}{2} v'''(x) \tag{41}$$

$$(\delta + m)d(x) = \kappa + m - \left(m + \mu - \sigma^2\right) x d'(x) + \frac{\sigma^2 x^2}{2} d''(x). \tag{42}$$

B.5.12 Properties of v and d at Points with Non-Smooth Debt Trading

Lumpy debt issuances and repurchases We now study lumpy debt trading decisions and show that at a debt-to-income ratio $\hat{x}$ at which the government ends up following such lumpy trading decision, v must be $\mathcal{C}^2$ and we must have $d'(\hat{x}) = v''(\hat{x}) = 0$.

Without loss of generality, consider a lumpy issuance. Consider $x_0 < \hat{x}$, where $\hat{x}$ is the point at which the government ends up after the lumpy issuance, and x_0 is such that at time t, starting at a debt-to-income level $x_t \in [x_0, \hat{x}]$, the government jumps to $x_{t+} = \hat{x}$. We must have $d(x) = d(\hat{x}) = d(x_0)$ for all $x \in [x_0, \hat{x}]$ thanks to creditor's rational expectation. It must then exist $x_2 > \hat{x}$ such that x_t evolves continuously on $[\hat{x}, x_2]$, otherwise $x_{t+} \neq \hat{x}$. $\hat{x}$ must then be a reflecting barrier when the process x_t starts at $x \geq \hat{x}$. In that case, equation (40) must hold for all $x \in (\hat{x}, x_2)$. We must also have $d'(\hat{x}) = 0$. Indeed, $d'(\hat{x}-) = 0$ since $d(x) = d(\hat{x})$ for all $x \in [x_0, \hat{x}]$, and $d'(\hat{x}+) = 0$ given the reflecting barrier. Finally, for $x \in (x_0, \hat{x})$, d is constant and v is an affine function, meaning that $v''(x) = 0$ on that interval. Finally, it must be the case that v is $\mathcal{C}^2$ at $x = \hat{x}$. Indeed, since v is convex, if for some reason v was not $\mathcal{C}^2$ at $x = \hat{x}$, this would mean that $v''(\hat{x}+) > v''(\hat{x}-) = 0$. But since $v'(x) + d(x) = 0$, differentiating this equation and evaluating it at $x = \hat{x}+$ leads to $v''(\hat{x}+) + d'(\hat{x}+) = v''(\hat{x}+) > 0$, where we have used the fact that $d'(\hat{x}+) = 0$. This is a contradiction, since we must have $v''(\hat{x}+) + d'(\hat{x}+) = 0$. Thus $v''(\hat{x}+) = v''(\hat{x}-) = 0$ and v is $\mathcal{C}^2$ at $x = \hat{x}$. A similar reasoning (omitted here) can be used in connection with lumpy debt buy-backs. Thus, on any jump interval $[x_0, \hat{x}]$, the value function v is affine in x, the debt price d is constant and strictly positive, v is $\mathcal{C}^2$ and d is $\mathcal{C}^1$ at the arrival point $x = \hat{x}$.

Isolated point of singular trading intensity In this section, we show that at any singular trading intensity point $\hat{x}$, the value function v is $\mathcal{C}^2$ and we must have $d'(\hat{x}) = v''(\hat{x}) = 0$. Consider such a point $\hat{x}$ of singular trading intensity. This means that on an open ball around $\hat{x}$, the state variable x_t satisfies

$$x_t^{(\gamma)} = x_0 + \int_0^t \left(\gamma_u - \left(m + \mu - \sigma^2\right) x_u^{(\gamma)}\right) du - \int_0^t \sigma x_u^{(\gamma)} dB_u + \int_0^t (2p-1)\, d\gamma_u^{(\hat{x})} \left(x_u^{(\gamma)}\right)$$

$$\gamma_t^{(\hat{x})} \left(x_t^{(\gamma)}\right) := \lim_{\epsilon \searrow 0} \frac{1}{2\epsilon} \int_0^t 1_{\{\hat{x}-\epsilon < x_t^{(\gamma)} < \hat{x}+\epsilon\}} ds$$

The probability $p \in [0,1], p \neq 1/2$ is the probability of "moving to the right", and it must satisfy

$$p d'\left(\hat{x}+\right) = (1-p) d'\left(\hat{x}-\right). \tag{43}$$

For more technical details, see the concept of "skew Brownian motion" in Harrison and Shepp (1981). The point $\hat{x}$ must be isolated, which means that on a left and right neighborhood of $\hat{x}$, equation (40) must hold. This means in particular that $v''(\hat{x}+) = v''(\hat{x}-)$, which means that $d'(\hat{x}+) = d'(\hat{x}-)$. Thus, in order for equation (43) to hold, it must be the case that $d'(\hat{x}+) = d'(\hat{x}-) = 0$, which must mean that v'' is continuous at $\hat{x}$, with $v''(\hat{x}) = 0$.

B.5.13 No Debt Welfare equals Autarky Value

We are now going to show that in any equilibrium, the sovereign does not achieve gains from trade when not indebted. Said another way, we are going to show that $v(0) = 1/(\delta - \mu)$, the autarky value. We will also show that in any equilibrium, the debt price is always weakly less than $\frac{\kappa+m}{\delta+m}$. Consider $x = 0$ – i.e. when the government is not indebted. Three cases can then arise.

- If the financing policy is absolutely continuous at $x = 0$, equations (41) and (42) hold, and if we take the limit of such equations when $x \to 0$, we obtain

$$v(0) = \frac{1}{\delta - \mu} \qquad\qquad d(0) = \frac{\kappa + m}{\delta + m}$$

- Imagine instead that at $x = 0$, the optimal financing policy is an impulse control, from $x = 0$ to $\hat{x}$. In such case, we know that the value function is linear on $x \in [0, \hat{x}]$ and we know that the debt price is constant on such interval. We also know, from the previous section, that we must have $v''(\hat{x}) = d'(\hat{x}) = 0$. For $x > \hat{x}$ and in the neighbourhood of $\hat{x}$, v must be strictly convex, i.e. $v''(x) > 0$, meaning that we must have $v'''_+(\hat{x}) \geq 0$. Thus, if one were to evaluate equation (42) at $x = \hat{x}+$, we would obtain

$$(\delta + m) d(\hat{x}) = \kappa + m + \frac{\sigma^2 \hat{x}^2}{2} d''(\hat{x}) = \kappa + m - \frac{\sigma^2 \hat{x}^2}{2} v'''_+(\hat{x}) \leq \kappa + m$$

In other words, in such case, we have

$$d(\hat{x}) = d(0) \leq \frac{\kappa + m}{\delta + m}$$

Since equation (40) holds in the (right) neighbourhood of $\hat{x}$, we must have

$$\begin{aligned}(\delta - \mu)\, v(\hat{x}) &= 1 - (\kappa + m)\hat{x} - (\mu + m)\, \hat{x} v'(\hat{x}) \\ &= 1 - (\kappa + m)\hat{x} + (\mu + m)\, \hat{x} d(\hat{x})\end{aligned}$$

But since the optimal policy is assumed to be an impulse control, it means that $v(0) = v(\hat{x}) + \hat{x}d(\hat{x})$, which we can use in the previous equation to obtain:

$$(\delta - \mu)\, v(0) = 1 + (\delta + m)\hat{x} \left[d\,(\hat{x}) - \frac{\kappa + m}{\delta + m} \right] \leq 1$$

In other words, it must be the case that $v(0) \leq 1/(\delta - \mu)$. But of course we know that the value function must be bounded below by the autarky value, which means that we must have $v(0) = 1/(\delta - \mu)$.

- Lastly, imagine that at $x = 0$, the optimal financing policy is singular control. Using the results from the previous section, this means that there exists a decreasing sequence $\{x_n\}_{n \geq 0}$ so that $x_n \to 0$ and for each n, $v''(x_n) = d'(x_n) = 0$. Since v is convex, for each n, it must be the case that on the right neighborhood of x_n, $v'''_+(x_n) = -d''_+(x_n) \geq 0$. Thus, evaluating (42) at each x_n, since d is decreasing, $d(x_n)$ converges monotonically to a limit that is less than or equal to $(\kappa + m)/(\delta + m)$. Finally, evaluating (40) at each x_n, one can use a reasoning similar to the previous section to conclude that $v(x_n)$ converges to a limit that is less than or equal to $1/(\delta - \mu)$ – and thus equal to $1/(\delta - \mu)$, since the value function needs to be weakly greater than the no-trade value.

The reasoning above also shows that $d(0) \leq \frac{\kappa+m}{\delta+m}$. Since d is decreasing, it then means that $d(x) \leq \frac{\kappa+m}{\delta+m}$ for all x. Thus, we have established that the value function for a government without debt is equal to the autarky value.

B.5.14 No Singular or Impulse Control in Equilibrium Financing Strategy

We end our proof by showing that in any equilibrium, the government always uses an absolutely continuous face value process, which leads to the conclusion that the equilibrium must be unique, and that it must correspond to the equilibrium we characterized in the main text. As a preliminary calculation, note $z(x)$ the value (per unit of income) if a no-debt government was immediately jumping from $x = 0$ to $x > 0$:

$$z(x) = v(x) + xd(x) = v(x) - xv'(x)$$

Note that $z'(x) = -xv''(x) \leq 0$, which means that z is decreasing, and that

$$z(x) \leq z(0) = \frac{1}{\delta - \mu}$$

Now let us prove that there can be no jump or points of singularity in our equilibrium. By way of contradiction, imagine $\hat{x}$ was a point of singular issuance intensity or arrival point of an issuance jump. We need to consider three cases:

- Imagine a jump occurs immediately when the sovereign has no debt. Let $\hat{x}$ be the arrival point of such initial jump. Then we have, for $x \le \hat{x}$

$$\frac{1}{\delta-\mu} - \frac{\kappa+m}{\delta+m}x \le v(x) = v(\hat{x}) + (\hat{x}-x)\,d(\hat{x}) = z(\hat{x}) - xd(\hat{x}),$$

  where the first inequality comes from the fact that v is bounded from below by the no-default, no-trade value, and where the second equality comes from the fact that the government jumps from any point $x \le \hat{x}$ to $\hat{x}$. But since $d(\hat{x}) \le \frac{\kappa+m}{\delta+m}$, this means that $\frac{1}{\delta-\mu} \le z(\hat{x})$, which is a contradiction.

- Imagine instead that a jump occurs between $x_1 > 0$ and $x_2 > x_1$; x_2 is the arrival debt-to-income ratio, and we know from previous sections that we must then have $v''(x_2) = 0$. Since we have a jump between x_1 and x_2, this also means that $v(x_1) = v(x_2) + (x_1 - x_2)v'(x_1)$. Consider the HJB equation satisfied by v at both x_1- and x_2+:

$$(\delta-\mu)v(x_1) = 1 - (\kappa+m)x_1 - (\mu+m)x_1 v'(x_1) + \frac{\sigma^2 x_1^2}{2} v''(x_1-)$$
$$(\delta-\mu)v(x_2) = 1 - (\kappa+m)x_2 - (\mu+m)x_2 v'(x_2)$$

  Note that the second order derivative v'' does not appear for the second HJB since $v''(x_2+) = 0$. Take the difference between those two HJBs, use $v(x_1) = v(x_2) + (x_1 - x_2)v'(x_1)$ and $v'(x_1) = v'(x_2)$ to obtain

$$(\delta-\mu)\,(x_1-x_2)\,v'(x_1) = -(\kappa+m)(x_1-x_2) - (\mu+m)v'(x_1)(x_1-x_2) + \frac{\sigma^2 x_1^2}{2} v''(x_1-)$$

  After simplification and division by $(x_2 - x_1)$ we obtain

$$d(x_1) = -v'(x_1) = \frac{\kappa+m}{\delta+m} + \frac{\sigma^2 x_1^2 v''(x_1-)}{2(x_2-x_1)} \ge \frac{\kappa+m}{\delta+m},$$

  where the inequality comes from the fact that $v''(x_1-) \ge 0$. But since $x_1 > 0$ and since the financing strategy for the government is absolutely continuous on an non-empty interval in the left neighborhood of x_1, it must be the case that $d(x_1) < \frac{\kappa+m}{\delta+m}$, thus we have a contradiction.

- Finally, let us assume that $\hat{x} > 0$ is a point of singular trading intensity. We then know that $v''(\hat{x}) = d'(\hat{x}) = 0$. Since d is decreasing and $d'(\hat{x}-) \le 0$, it must be the case that $d''(\hat{x}-) \ge 0$. Consider equation (42), satisfied by d at $\hat{x}-$:

$$\begin{aligned}(\delta+m)d(\hat{x}) &= \kappa + m - (\mu+m-\sigma^2)\hat{x}d'(\hat{x}-) + \frac{\sigma^2\hat{x}^2}{2}d''(\hat{x}-)\\ &= \kappa+m+\frac{\sigma^2\hat{x}^2}{2}d''(\hat{x}-)\\ &\ge \kappa+m,\end{aligned}$$

where the second equality uses $d'(\hat{x}) = 0$ and the last inequality stems from $d''(\hat{x}-) \geq 0$. Thus, $d(\hat{x}) \geq \frac{\kappa+m}{\delta+m}$, which is a contradiction – for the same reason as in the previous case.

All the cases considered above show that the equilibrium financing strategy of the government cannot feature any singular point of trading intensity, or any impulse control. Thus, the financing strategy of the government must be absolutely continuous, and the MPE must be unique. □

B.6 Ergodic Distribution and Average Default Rate

The drift rate $\mu_x(x)$ of the state variable x and the volatility $\sigma_x(x)$ are equal to:

$$\begin{aligned}\mu_x(x) :&= g(x) - \left(m + \mu - |\sigma|^2\right) x \\ &= \left[\left(\frac{\delta - r}{\xi - 1}\right)\left(\frac{1 - \alpha\theta^{\xi}}{1 - \alpha\theta}\right)\left(\frac{\bar{x}}{x}\right)^{\xi - 1} - \left(m + \mu + \boldsymbol{\nu}\cdot\boldsymbol{\sigma} - |\boldsymbol{\sigma}|^2 + \frac{\delta - r}{\xi - 1}\right)\right] x \\ \sigma_x(x) :&= \sigma x\end{aligned}$$

The stationary distribution f of the state variable x_t under the measure $\mathbb{P}$ solves the Kolmogorov-Forward equation:

$$0 = \frac{dJ}{dx} = -\frac{d}{dx}\left[\mu_x(x)f(x)\right] + \frac{d^2}{dx^2}\left[\frac{\sigma_x^2(x)}{2}f(x)\right] \tag{44}$$

B.6.1 Is the boundary $x = 0$ ever reached?

What is the behavior of x_t at the boundary $x = 0$? To understand this question better, we need to determine whether such boundary is a regular, entrance, exit or natural boundary. Thus, define the scale density by:

$$\omega(x; x_0) := \exp\left[-\int_{x_0}^{x} \frac{2\mu_x(t)}{|\sigma_x(t)|^2}dt\right] = \left(\frac{x}{x_0}\right)^{b} \exp\left(\frac{-a}{1-\xi}\left(x^{1-\xi} - x_0^{1-\xi}\right)\right),$$

with the constant $a > 0$ and b defined via

$$a := \frac{\delta - r}{\xi - 1}\left(\frac{1 - \alpha\theta^{\xi}}{1 - \alpha\theta}\right)\frac{2\bar{x}^{\xi - 1}}{|\sigma|^2} \qquad b := \frac{2}{|\sigma|^2}\left(\frac{\delta - r}{\xi - 1} + m + \mu + \boldsymbol{\nu}\cdot\boldsymbol{\sigma} - |\boldsymbol{\sigma}|^2\right)$$

Compute the following:

$$A(x^*) := \int_{x^*}^{x_0}\left(\int_{x^*}^{t}\omega(z; x_0)dz\right)\frac{2}{\omega(t; x_0)|\sigma_x(t)|^2}dt; \qquad B(x^*) := \int_{x^*}^{x_0}\left(\int_{t}^{x^*}\omega(z; x_0)dz\right)\frac{2}{\omega(t; x_0)|\sigma_x(t)|^2}dt$$

The boundary $x^* \in \{0, \bar{x}\}$ is classified depending on the finiteness of these integrals (see short presentation of Feller's classification in Durrett and Durrett (2008)):

1. x^* is *regular* if $A(x^*) < +\infty$ and $B(x^*) < +\infty$
2. x^* is *entrance* if $A(x^*) = +\infty$ and $B(x^*) < +\infty$

3. x^* is *exit* if $A(x^*) < +\infty$ and $B(x^*) = +\infty$

4. x^* is *natural* if $A(x^*) = +\infty$ and $B(x^*) = +\infty$

Entrance and natural boundaries are inaccessible, meaning they are never reached; regular and exit boundaries are accessible. Clearly, the boundary $\bar{x}$ is regular and thus accessible. In addition, $\omega(x; x_0) \to +\infty$ as $x \to 0$. This means that the quantities $A(0)$ and $B(0)$ are not finite: $x = 0$ is a natural boundary, and is thus not accessible. Said differently, an indebted government can never entirely repay back its entire existing stock of debt once it has issued some debt.

B.6.2 Solution to the KF equation

The ergodic density satisfies $dJ/dx = 0$, where $J(x) = \mu_x(x)f(x) - \frac{1}{2}\frac{d}{dx}\left(\sigma_x^2(x)f(x)\right)$. We can integrate this equation on $[\theta\bar{x}, \bar{x}]$ to obtain $J(x) = \lambda_d$, where $\lambda_d := J(\bar{x})$ is the ergodic default rate. Thus, for $x \in [\theta\bar{x}, \bar{x}]$, f satisfies

$$\mu_x(x)f(x) - \frac{d}{dx}\left[\frac{\sigma_x(x)^2}{2}f(x)\right] = \lambda_d$$

With the boundary condition $f(\bar{x}) = 0$ (since $\bar{x}$ is an exist point), the solution to this first order ODE is

$$f(x) = \lambda_d \int_x^{\bar{x}} \exp\left(-\int_x^u \frac{2\mu_x(s)}{\sigma_x^2(s)}ds\right) du = \lambda_d \int_x^{\bar{x}} \omega(u; x)du$$

For $x \in (0, \theta\bar{x}]$, the probability flux is constant equal to zero since $x = 0$ is unattainable. Thus,

$$\mu_x(x)f(x) - \frac{d}{dx}\left[\frac{\sigma_x(x)^2}{2}f(x)\right] = 0$$

Given that f is continuous at $x = \theta\bar{x}$, the solution to this first order ODE is

$$f(x) = \lambda_d \int_{\theta\bar{x}}^{\bar{x}} \omega(u; x)du$$

□

B.7 GDP-Linked Bonds

The bonds issued by the government are now GDP-linked, with weighting vector $\boldsymbol{\varsigma}$, such that the debt face value F_t follows:

$$F_t^{(G,\tau)} = \int_0^t \left(G\left(Y_u^{(\tau)}, F_u^{(G,\tau)}, s_u\right) - mF_u^{(G,\tau)}\right) du + \int_0^t F_u^{(G,\tau)}\boldsymbol{\varsigma} \cdot d\boldsymbol{B}_u + \int_0^t (\theta\alpha - 1)\, F_{u-}^{(G,\tau)} dN_{d,u}^{(\tau)}.$$

The value function V is written $V(Y, F) = Yv(x)$. Under Pr and $\tilde{\Pr}$, the debt-to-income ratio follows:

$$dx_t^{(g,\tau)} = \left(g(x_t^{(g,\tau)}, s_t) - (m + \mu - \boldsymbol{\sigma} \cdot (\boldsymbol{\sigma} - \boldsymbol{\varsigma}))\, x_t^{(g,\tau)}\right) dt - x_t^{(g,\tau)}\, (\boldsymbol{\sigma} - \boldsymbol{\varsigma}) \cdot d\boldsymbol{B_t} + (\theta - 1)\, x_t^{(g,\tau)} dN_{d,t}^{(\tau)}$$
$$dx_t^{(g,\tau)} = \left(g(x_t^{(g,\tau)}, s_t) - (m + \mu)\, x_t^{(g,\tau)}\right) dt - x_t^{(g,\tau)}\, (\boldsymbol{\sigma} - \boldsymbol{\varsigma}) \cdot d\tilde{\boldsymbol{B}}_t + (\theta - 1)\, x_t^{(g,\tau)} dN_{d,t}^{(\tau)}.$$

In the continuation region $(0, \bar{x})$, v satisfies:

$$(\delta - \mu)\, v(x) = 1 - (\kappa + m)\, x - (\mu + m)\, x v'(x) + \frac{1}{2}|\boldsymbol{\sigma} - \boldsymbol{\varsigma}|^2 x^2 v''(x).$$

This is a second order ordinary differential equation, whose general solutions are power functions of x. The exponent of the general solutions solves the quadratic equation:

$$\frac{1}{2}|\boldsymbol{\sigma} - \boldsymbol{\varsigma}|^2 \xi_{\varsigma}^2 - \left(m + \mu + \frac{1}{2}|\boldsymbol{\sigma} - \boldsymbol{\varsigma}|^2\right) \xi_{\varsigma} - (\delta - \mu) = 0.$$

This quadratic equation admits one positive, and one negative roots. We also know that $\xi_{\varsigma} > 1$. Since v must be finite as $x \to 0$, we eliminate the negative root, and note ξ_{ς} the positive one. Our second boundary condition uses the fact that upon default, the small open economy suffers a discrete income drop by a factor α, and immediately emerges from autarky with a debt-to-income ratio that is a fraction θ of its pre-default debt-to-income ratio:

$$v(\bar{x}_{\varsigma}) = \alpha v(\theta \bar{x}_{\varsigma}).$$

Using these, we can express v as follows on $[0, \bar{x}]$:

$$v(x) = \frac{1}{\delta - \mu}\left[1 - \left(\frac{1-\alpha}{1-\alpha\theta^{\xi_{\varsigma}}}\right)\left(\frac{x}{\bar{x}_{\varsigma}}\right)^{\xi_{\varsigma}}\right] - \left(\frac{\kappa + m}{\delta + m} x\right)\left[1 - \left(\frac{1-\alpha\theta}{1-\alpha\theta^{\xi_{\varsigma}}}\right)\left(\frac{x}{\bar{x}_{\varsigma}}\right)^{\xi_{\varsigma}-1}\right].$$

The smooth-pasting condition takes the usual form:

$$v'(\bar{x}_{\varsigma}) = \alpha\theta v'(\theta \bar{x}_{\varsigma}).$$

Collecting these together, we compute the following default boundary $\bar{x}_{\varsigma}$:

$$\bar{x}_{\varsigma} = \frac{\xi_{\varsigma}}{\xi_{\varsigma} - 1}\left(\frac{\delta + m}{\kappa + m}\right)\left(\frac{1-\alpha}{1-\alpha\theta}\right)\frac{1}{\delta - \mu}.$$

The debt price d per unit of debt outstanding can be computed by leveraging the first order condition $d(x) = -v'(x)$. In other words, for $x \in [0, \bar{x}_{\varsigma}]$, we have:

$$d(x) = \left(\frac{\kappa + m}{\delta + m}\right)\left[1 - \left(\frac{1-\alpha\theta}{1-\alpha\theta^{\xi_{\varsigma}}}\right)\left(\frac{x}{\bar{x}_{\varsigma}}\right)^{\xi_{\varsigma}-1}\right].$$

Note that in the continuation region, the value function v takes the following form:

$$v(x) = \frac{1}{\delta - \mu}\left(1 - \left(\frac{1-\alpha}{1-\alpha\theta^{\xi_{\varsigma}}}\right)\left(\frac{x}{\bar{x}_{\varsigma}}\right)^{\xi_{\varsigma}}\right) - x d(x).$$

The required expected excess return on the sovereign debt can be easily computed:

$$\pi(x, s) = -\frac{x d'(x)}{d(x)}\,(\boldsymbol{\sigma} - \boldsymbol{\varsigma}) \cdot \boldsymbol{\nu}(s) = \frac{\xi_{\varsigma} - 1}{\left(\frac{1-\alpha\theta^{\xi_{\varsigma}}}{1-\alpha\theta}\right)\left(\frac{\bar{x}_{\varsigma}}{x}\right)^{\xi_{\varsigma}-1} - 1}\,(\boldsymbol{\sigma} - \boldsymbol{\varsigma}) \cdot \boldsymbol{\nu}(s).$$

The issuance policy now takes a different form. Indeed, note that the debt price satisfies:

$$\begin{aligned} d(x) &= \mathbb{E}^{\mathbb{Q}}_{x,s}\left[\int_0^\infty e^{-\int_0^t \left(r(s_u)+m+\frac{1}{2}|\boldsymbol{\varsigma}|^2\right)du+\int_0^t \boldsymbol{\varsigma}\cdot d\boldsymbol{B}_u}\,(\alpha\theta)^{N^{(\tau)}_{d,t}}\,(\kappa+m)dt\right] \\ &= \mathring{\mathbb{E}}_{x,s}\left[\int_0^\infty e^{-\int_0^t (r(s_u)+m+\boldsymbol{\varsigma}\cdot\boldsymbol{\nu}(s_u))du}\,(\alpha\theta)^{N^{(\tau)}_{d,t}}\,(\kappa+m)dt\right]. \end{aligned}$$

In the above, we have introduced the measure $\mathring{\Pr}$, defined for any arbitrary Borel set $A \subseteq \mathcal{F}_t$ via $\mathring{\Pr}(A) = \mathbb{E}\left[\exp\left(-\frac{|\boldsymbol{\varsigma}|^2}{2}t+\boldsymbol{\varsigma}\cdot\tilde{\boldsymbol{B}}_t\right)A\right]$. $\mathring{\boldsymbol{B}}_t := \boldsymbol{B}^{\mathbb{Q}}_t - \boldsymbol{\varsigma}t$ is a standard Brownian motion under $\mathring{\Pr}$, and under such measure the debt-to-income ratio follows:

$$\begin{aligned} dx^{(g,\tau)}_t = \left(g(x^{(g,\tau)}_t, s_t) - \left(m+\mu-|\boldsymbol{\sigma}-\boldsymbol{\varsigma}|^2-\boldsymbol{\nu}(s)\cdot(\boldsymbol{\sigma}-\boldsymbol{\varsigma})\right)x^{(g,\tau)}_t\right)dt \\ - x^{(g,\tau)}_t\,(\boldsymbol{\sigma}-\boldsymbol{\varsigma})\cdot d\mathring{\boldsymbol{B}}_t + (\theta-1)\,x^{(g,\tau)}_t dN^{(\tau)}_{d,t}. \end{aligned}$$

The debt price thus satifies the following Feynman-Kac equation:

$$(r(s)+m+\boldsymbol{\varsigma}\cdot\boldsymbol{\nu}(s))\,d(x) = \kappa+m+\left[g(x,s)-\left(m+\mu-|\boldsymbol{\sigma}-\boldsymbol{\varsigma}|^2-\boldsymbol{\nu}(s)\cdot(\boldsymbol{\sigma}-\boldsymbol{\varsigma})\right)x\right]d'(x)+\frac{|\boldsymbol{\sigma}-\boldsymbol{\varsigma}|^2}{2}x^2 d''(x).$$

As usual, one can use this equation to back out the issuance policy:

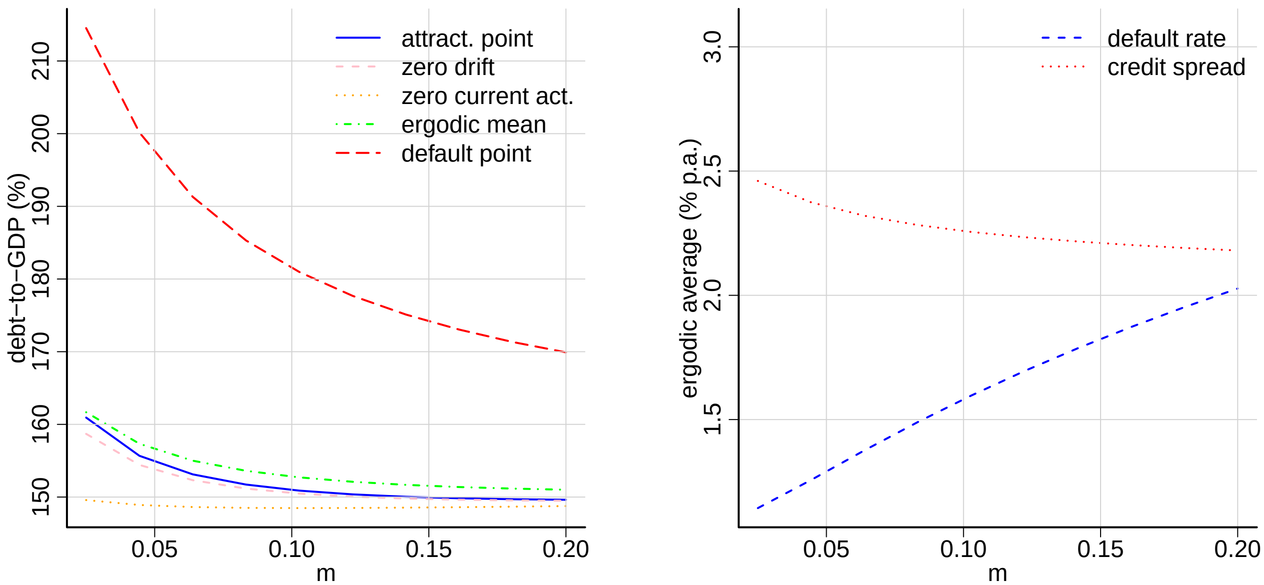
$$\begin{aligned} g^*(x,s) &= \frac{d(x)}{-d'(x)}(\delta - r(s) - \boldsymbol{\varsigma}\cdot\boldsymbol{\nu}(s)) - x\,(\boldsymbol{\sigma}-\boldsymbol{\varsigma})\cdot\boldsymbol{\nu}(s) \\ &= \frac{\delta-(r(s)+\boldsymbol{\varsigma}\cdot\boldsymbol{\nu}(s))}{\xi_{\varsigma}-1}\left[\left(\frac{1-\alpha\theta^{\xi_\varsigma}}{1-\alpha\theta}\right)\left(\frac{\bar{x}_\varsigma}{x}\right)^{\xi_\varsigma-1}-1\right]x - x\,(\boldsymbol{\sigma}-\boldsymbol{\varsigma})\cdot\boldsymbol{\nu}(s). \end{aligned}$$

□

B.8 Comparative Statics – Numerical Results

Figure 3: Sensitivities to m

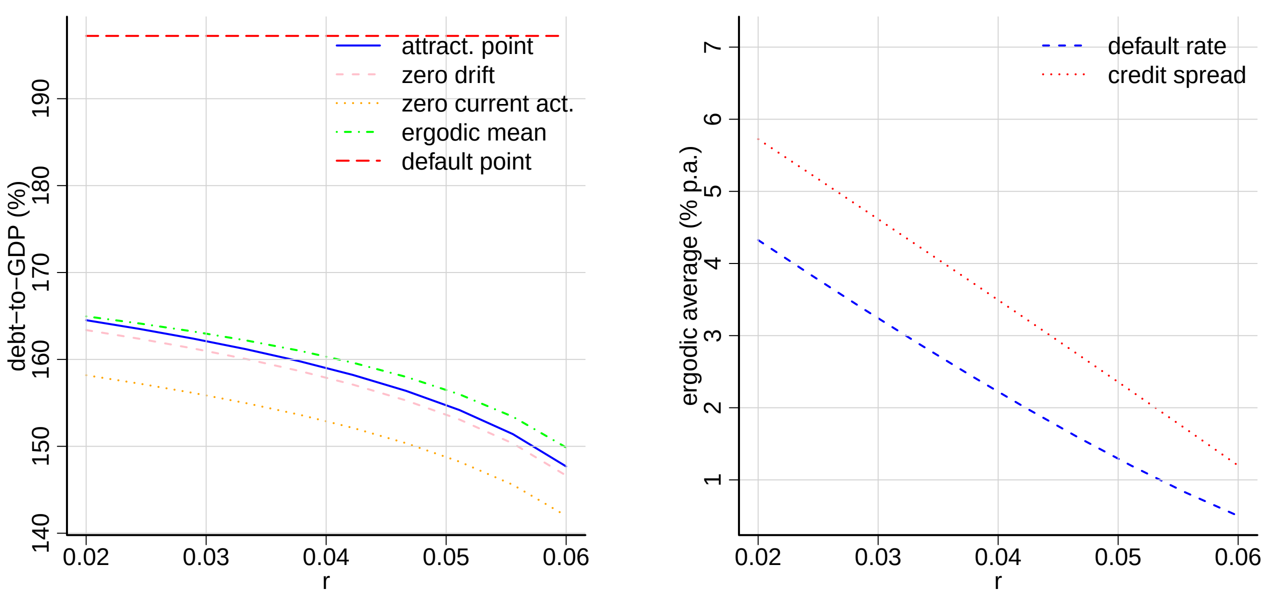
(a): debt-to-income **(b):** Default rate and credit spreads



Plots computed assuming $\mu = 2\%$ p.a., $\sigma = 10\%$ p.a., $1/m = 20$ years, $\theta = 50\%$, $\alpha = 96\%$, $\nu = 40\%$, $r = \kappa = 5\%$ and $\delta = 7\%$.

Figure 4: Sensitivities to r

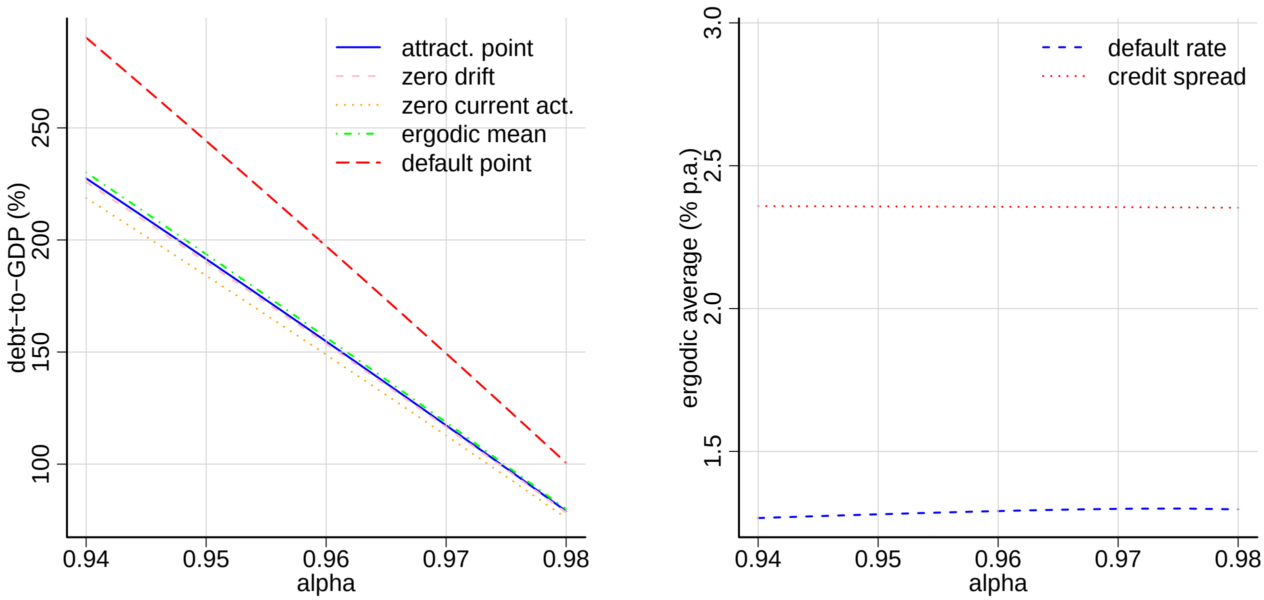
(a): debt-to-income **(b):** Default rate and credit spreads



Plots computed assuming $\mu = 2\%$ p.a., $\sigma = 10\%$ p.a., $1/m = 20$ years, $\theta = 50\%$, $\alpha = 96\%$, $\nu = 40\%$, $r = \kappa = 5\%$ and $\delta = 7\%$.

Figure 5: Sensitivities to α

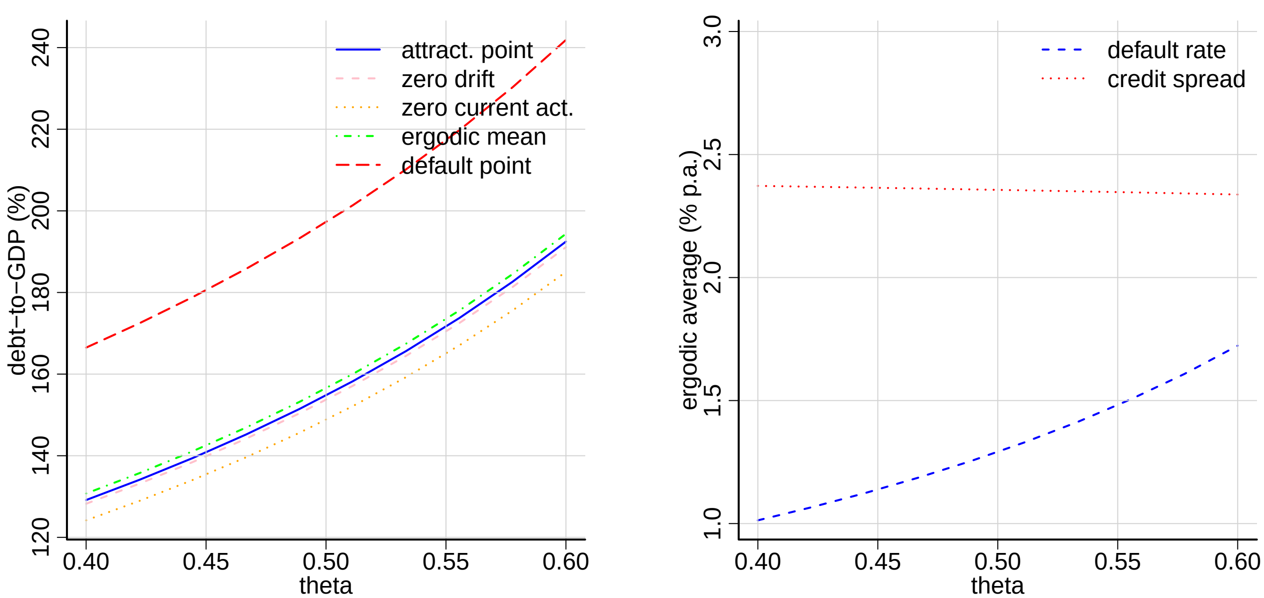
(a): debt-to-income **(b):** Default rate and credit spreads



Plots computed assuming $\mu = 2\%$ p.a., $\sigma = 10\%$ p.a., $1/m = 20$ years, $\theta = 50\%$, $\alpha = 96\%$, $\nu = 40\%$, $r = \kappa = 5\%$ and $\delta = 7\%$.

Figure 6: Sensitivities to θ

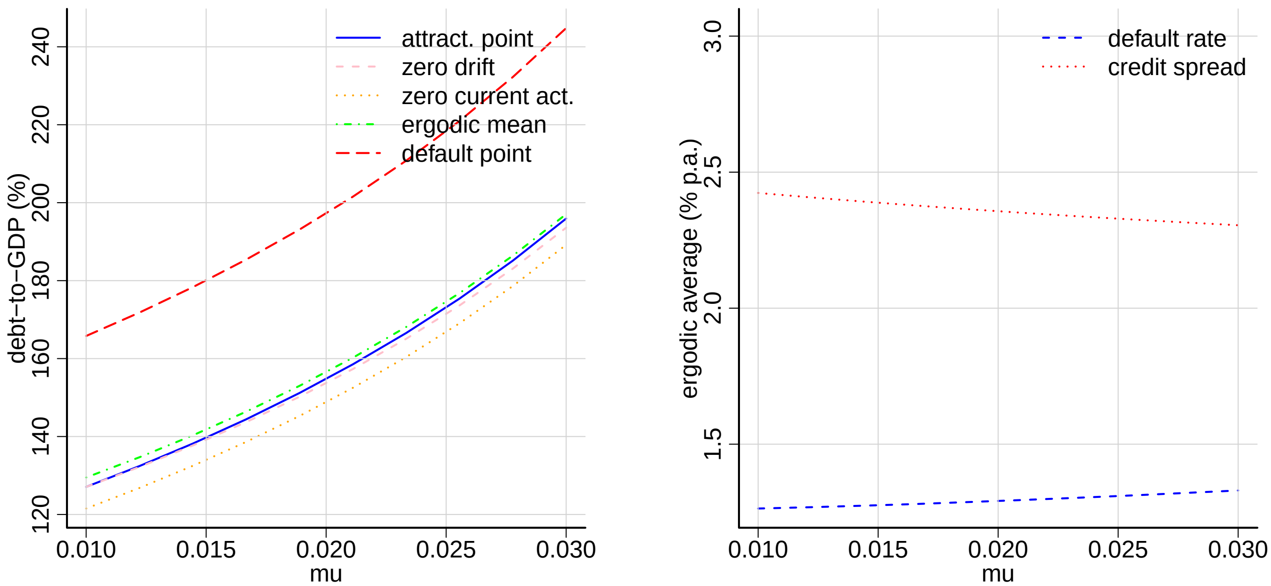
(a): debt-to-income **(b):** Default rate and credit spreads



Plots computed assuming $\mu = 2\%$ p.a., $\sigma = 10\%$ p.a., $1/m = 20$ years, $\theta = 50\%$, $\alpha = 96\%$, $\nu = 40\%$, $r = \kappa = 5\%$ and $\delta = 7\%$.

Figure 7: Sensitivities to μ

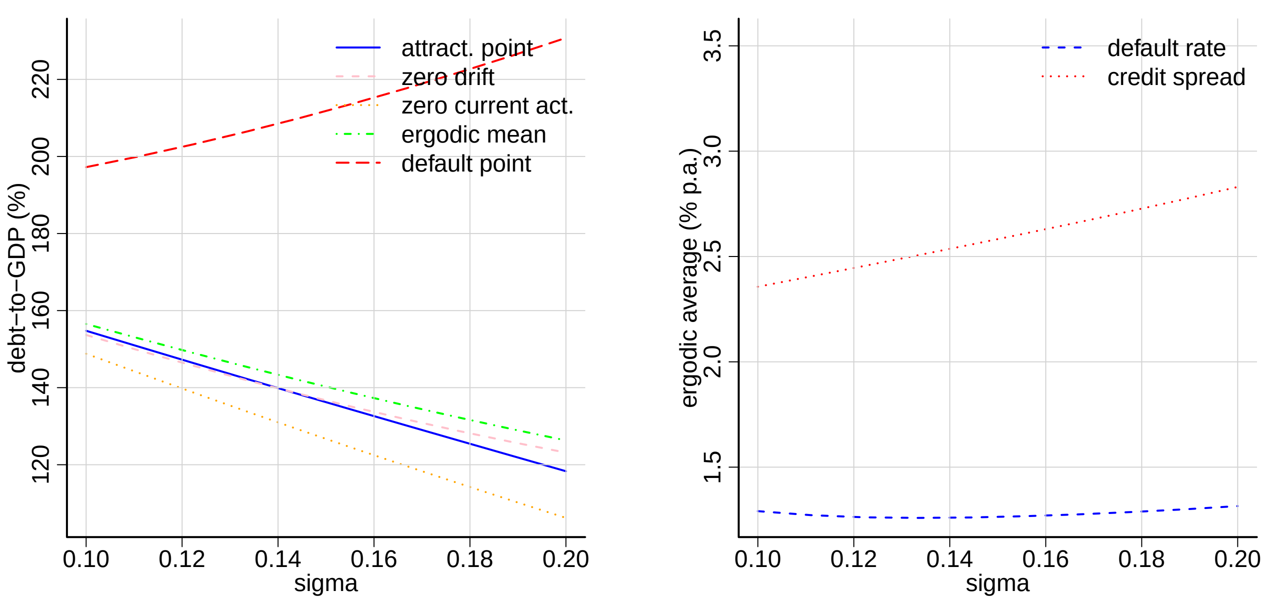
(a): debt-to-income | **(b):** Default rate and credit spreads



Plots computed assuming $\mu = 2\%$ p.a., $\sigma = 10\%$ p.a., $1/m = 20$ years, $\theta = 50\%$, $\alpha = 96\%$, $\nu = 40\%$, $r = \kappa = 5\%$ and $\delta = 7\%$.

Figure 8: Sensitivities to σ

(a): debt-to-income | **(b):** Default rate and credit spreads



Plots computed assuming $\mu = 2\%$ p.a., $\sigma = 10\%$ p.a., $1/m = 20$ years, $\theta = 50\%$, $\alpha = 96\%$, $\nu = 40\%$, $r = \kappa = 5\%$ and $\delta = 7\%$.

C Proofs for: Restoring Gains from Trade

C.1 Infrequent Trading Opportunities

In this section, the government can issue debt only at Poisson arrival times (parameter $1/\Delta$). We assume that in default, the government loses its entire income stream (i.e. $\alpha = 0$), while creditors lose their entire investment (i.e. $\theta = 0$). In the discussion that follows, we will assume the existence of an MPE characterized by the set of equations (45), (48) and (46), to be described shortly.

C.1.1 No Commitment Equilibrium

In an MPE, the scaled value function for the government satisfies (in the continuation region)

$$\left(\delta + \frac{1}{\Delta} - \mu\right) v_\Delta(x) = 1 - (\kappa + m)x - (\mu + m)\, x v'_\Delta(x) + \frac{\sigma^2 x^2}{2} v''_\Delta(x) + \frac{1}{\Delta} \max_y \left[v_\Delta(y) + (y - x) d_\Delta(y)\right]. \quad (45)$$

Equation (45) is identical to equation (19), except for the terms involving $1/\Delta$ and related to the infrequent debt rebalancing opportunities. Let $n_\Delta(x)$ be the "jump-to" state, in other words

$$n_\Delta(x) = \arg\max_y \left[v_\Delta(y) + (y - x) d_\Delta(y)\right]. \quad (46)$$

The "jump-to" state satisfies the optimality condition

$$v'_\Delta\left(n_\Delta(x)\right) + d_\Delta\left(n_\Delta(x)\right) + \left(n_\Delta(x) - x\right) d'_\Delta\left(n_\Delta(x)\right) = 0. \quad (47)$$

Optimality at the default boundary yields the standard smooth pasting condition $v'_\Delta(\bar{x}_\Delta) = 0$. The debt price satisfies the usual Feynman-Kac equation

$$\left(\frac{1}{\Delta} + r + m\right) d_\Delta(x) = \kappa + m - \left(\mu + m - \sigma^2\right) x d'_\Delta(x) + \frac{\sigma^2 x^2}{2} d''_\Delta(x) + \frac{1}{\Delta} d_\Delta\left(n_\Delta(x)\right). \quad (48)$$

At the default boundary, we must also have $d_\Delta(\bar{x}_\Delta) = 0$. If we differentiate equation (45) and use the envelop theorem, we obtain

$$\left(\frac{1}{\Delta} + \delta + m\right) v'_\Delta(x) = -(\kappa + m) - \left(\mu + m - \sigma^2\right) x v''_\Delta(x) + \frac{\sigma^2 x^2}{2} v'''_\Delta(x) - \frac{1}{\Delta} d_\Delta\left(n_\Delta(x)\right).$$

We add (48) to this latter equation, and obtain a differential equation for $h_\Delta(x) := v'_\Delta(x) + d_\Delta(x)$:

$$\left(\frac{1}{\Delta} + \delta + m\right) h_\Delta(x) = (\delta - r) d_\Delta(x) - \left(\mu + m - \sigma^2\right) x h'_\Delta(x) + \frac{\sigma^2 x^2}{2} h''_\Delta(x).$$

Combine this with the boundary condition $h_\Delta(\bar{x}_\Delta) = 0$, and we obtain

$$h_\Delta(x) = \mathbb{E}^{nt}_x \left[\int_0^\tau e^{-(\delta + m + 1/\Delta)s} \left(\delta - r\right) d_\Delta(x_s) ds\right], \quad (49)$$

with $\mathbb{E}^{nt}$ is the expectation operator under the no-trade policy, under which x_t satisfies

$$dx_t = -\left(m + \mu - \sigma^2\right) x_t dt - \sigma x_t dB_t.$$

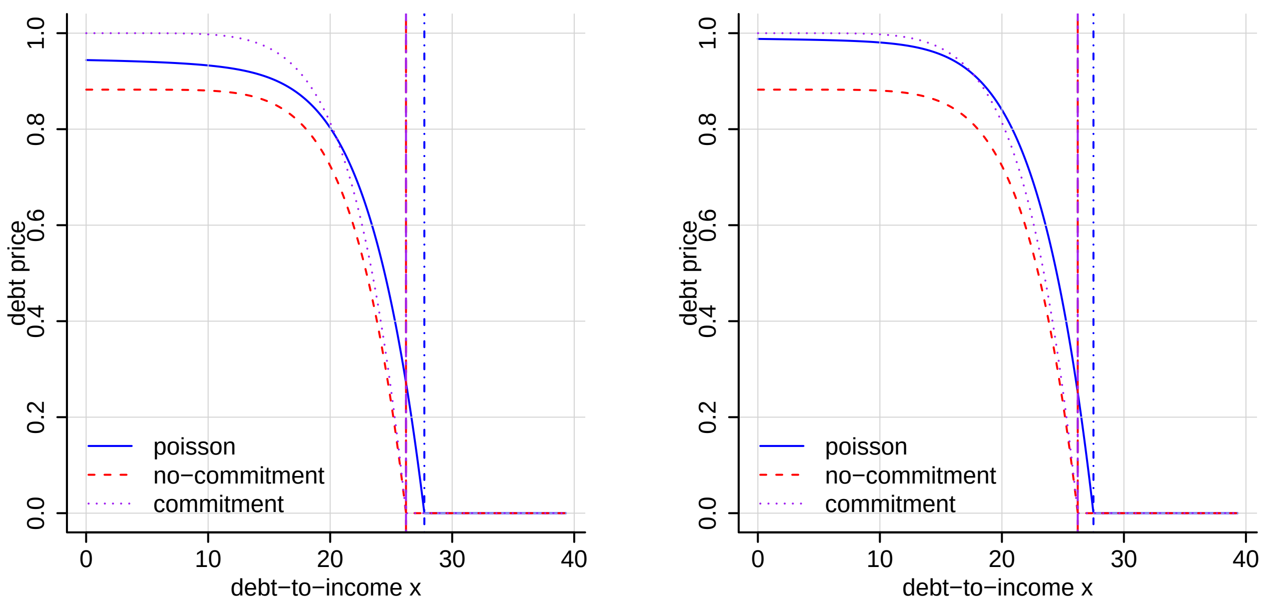
Note that $h_\Delta(x)$ represents the marginal gains from issuing one unit of debt. Equation (49) shows that $h_\Delta(x) > 0$; and because $d'_\Delta(\cdot) < 0$, we can use equation (47) to conclude that $n_\Delta(x) > x$. In other words, the government will issue some debt whenever it has the Poisson opportunity to adjust its debt balance. Finally, this also allows us to define the attraction point $x_{\Delta,a}$ – the debt-to-income ratio at which expected issuances equals redemptions:

$$\frac{1}{\Delta}\left(n_\Delta\left(x_{\Delta,a}\right) - x_{\Delta,a}\right) = m x_{\Delta,a}. \tag{50}$$

□

Figure 9: Debt price vs. Δ

(a): debt price with high Δ **(b):** debt price with low Δ

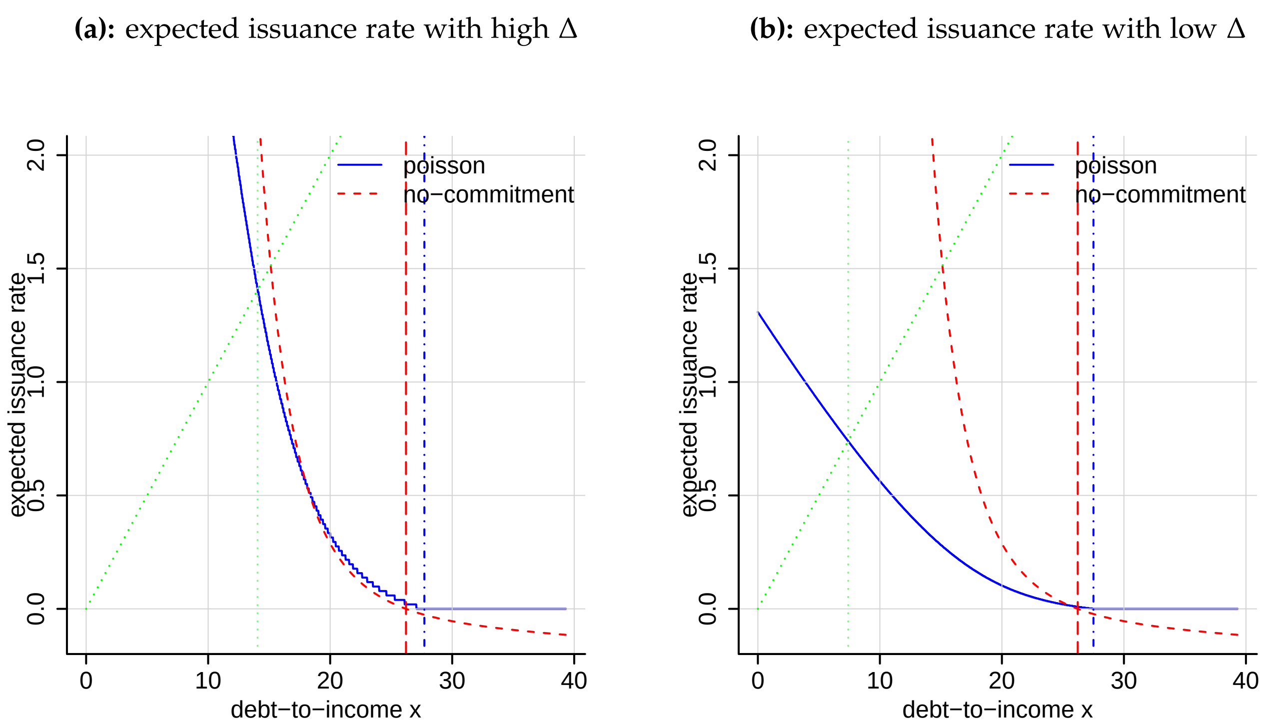


Plots show the debt price (blue solid line) for two different values of the expected trading interval Δ, equal to 10 (left hand side) and 1 (right hand side). We also show (red dotted line) the debt price in our no-commitment Smooth MPE, and (purple dotted line) the debt price in the equilibrium where the government can commit never to issue any debt. Plots computed assuming $\mu = 2\%$ p.a., $\sigma = 20\%$ p.a., $1/m = 10$ years, $\theta = 0\%$, $\alpha = 0\%$, $\nu = 0\%$, $r = \kappa = 5\%$ and $\delta = 7\%$.

C.1.2 Commitment Equilibrium

Imagine now that the government can commit to a particular debt strategy. In particular, we assume that at each debt financing opportunity, the government rebalances its debt towards a debt-to-income target x^*. At time zero, when the government is not indebted, it chooses the optimal debt-to-income x^* at which it commits to releverage, when given the opportunity to do so in the future. Note $v_{\Delta,c}\left(\cdot; x^*\right)$ (resp. $d_{\Delta,c}\left(\cdot; x^*\right)$) the scaled value function (resp. debt price) given our commitment assumption. In this

Figure 10: Expected issuance rate vs. Δ



Plots show (blue solid line) the expected issuance rate $(n_\Delta(x) - x)/\Delta$ for two different expected trading intervals Δ, equal to 10 years (left hand side) and 1 year (right hand side). We also show (red dotted line) the issuance rate $g(x)$ in our no-commitment Smooth MPE. Plots computed assuming $\mu = 2\%$ p.a., $\sigma = 20\%$ p.a., $1/m = 10$ years, $\theta = 0\%$, $\alpha = 0\%$, $\nu = 0\%$, $r = \kappa = 5\%$ and $\delta = 7\%$.

commitment MPE, the scaled value function for the government satisfies

$$\left(\frac{1}{\Delta} + \delta - \mu\right) v_{\Delta,c}(x; x^*) = 1 - (\kappa + m)x - (\mu + m)\, x v'_{\Delta,c}(x; x^*) + \frac{\sigma^2 x^2}{2} v''_{\Delta,c}(x; x^*) + \frac{1}{\Delta} \max\left[0, v_{\Delta,c}(x^*; x^*) + (x^* - x) d_{\Delta,c}(x^*; x^*)\right].$$

The maximum operator indicates that at the time the government has the opportunity to adjust its debt-to-income ratio to x^*, it can choose to default rather than change its indebtedness. In equilibrium, this may happen when the government has to buy back its own debt, and finds it more appealing to default. Note $\mathbb{J}_c(x; x^*) := \mathbb{1}\{v_{\Delta,c}(x^*; x^*) + (x^* - x) d_{\Delta,c}(x^*; x^*) \geq 0\}$ the "survival" indicator, equal to 1 whenever a trading opportunity occurs and the government prefers to change its debt-to-income rather than default. The debt price must then satisfy

$$\left(\frac{1}{\Delta} + r + m\right) d_{\Delta,c}(x; x^*) = \kappa + m - \left(\mu + m - \sigma^2\right) x d'_{\Delta,c}(x; x^*) + \frac{\sigma^2 x^2}{2} d''_{\Delta,c}(x; x^*) + \frac{1}{\Delta} \mathbb{J}_c(x; x^*) d_{\Delta,c}(x^*; x^*).$$

It is thus natural to postulate an equilibrium with 2 default boundaries $\bar{x}_c(x^*)$ and $\bar{\bar{x}}_c(x^*) > \bar{x}_c(x^*) > x^*$.

- When $x \leq \bar{x}_c(x^*)$, the debt-to-income ratio evolves with Brownian shocks only, except at Poisson arrival times, points at which the government issues (or buys back) a lump amount of debt to reach the debt-to-income ratio x^*;

- When $x \in (\bar{x}_c(x^*), \bar{\bar{x}}_c(x^*))$, the debt-to-income ratio evolves with Brownian shocks only, except at Poisson arrival times, points at which the government finds it optimal to default (rather than to buy back a lump amount of debt to reach the debt-to-income ratio x^*);
- When x reaches $\bar{\bar{x}}_c(x^*)$ (or even above), the government elects to default.

At $\bar{x}_c(x^*)$, the value function and debt price must be $\mathcal{C}^1$. At the default boundary $\bar{\bar{x}}_c(x^*)$, we have the following value matching and smooth pasting conditions for the debt price and value function:

$$d_{\Delta,c}\left(\bar{\bar{x}}_c(x^*); x^*\right) = 0 \qquad v_{\Delta,c}\left(\bar{\bar{x}}_c(x^*); x^*\right) = 0 \qquad v'_{\Delta,c}\left(\bar{\bar{x}}_c(x^*); x^*\right) = 0.$$

One can solve for the debt prices, using $\eta_2 > 0 > \eta_1$ for the roots of the characteristic polynomial

$$\frac{\sigma^2}{2}\eta^2 - \left(m + \mu - \frac{\sigma^2}{2}\right)\eta - (r + m + \frac{1}{\Delta}) = 0.$$

Denote $d^* := d\left(x^*; x^*\right)$, and $\bar{d} := d\left(\bar{x}_c(x^*); x^*\right)$. The debt price satisfies

$$d(x;x^*) = \frac{\kappa + m + d^*/\Delta}{r + m + 1/\Delta} - \left[\frac{\kappa + m + d^*/\Delta}{r + m + 1/\Delta} - \bar{d}\right]\left(\frac{x}{\bar{x}_c(x^*)}\right)^{\eta_2}, \qquad x \in (0, \bar{x}_c(x^*))$$

$$d(x;x^*) = \frac{\kappa + m}{r + m + 1/\Delta} + d_1\left(\frac{x}{\bar{\bar{x}}_c(x^*)}\right)^{\eta_1} + d_2\left(\frac{x}{\bar{\bar{x}}_c(x^*)}\right)^{\eta_2}, \qquad x \in \left(\bar{x}_c(x^*), \bar{\bar{x}}_c(x^*)\right).$$

Given $x^*, \bar{x}_c(x^*)$ and $\bar{\bar{x}}_c(x^*)$, we have 4 unknown constants $(d^*, \bar{d}, d_1, d_2)$ to determine. The requirement that $d(x^*;x^*) = d^*$ delivers one equation. The requirement that d be $\mathcal{C}^1$ at $x = \bar{x}_c(x^*)$ delivers two more equations. The requirement that $d(\bar{\bar{x}}_c(x^*);x^*) = 0$ delivers one last equation. Note that this system of 4 equations in 4 unknown is a linear system.

We can also solve for the value function. Let $\xi_2 > 0 > \xi_1$ be the roots of the characteristic polynomial

$$\frac{\sigma^2}{2}\xi^2 - \left(m + \mu + \frac{\sigma^2}{2}\right)\eta - (\delta + \frac{1}{\Delta} - \mu) = 0$$

The value function takes the following form

$$v_{\Delta,c}(x;x^*) = \left[\frac{1 + \left(v^* + x^* d^*\right)/\Delta}{\delta + 1/\Delta - \mu}\right]\left[1 - \left(\frac{x}{\bar{x}_c(x^*)}\right)^{\xi_2}\right] - x\left[\frac{\kappa + m + d^*/\Delta}{\delta + m + 1/\Delta}\right]\left[1 - \left(\frac{x}{\bar{x}_c(x^*)}\right)^{\xi_2 - 1}\right] + \bar{v}\left(\frac{x}{\bar{x}_c(x^*)}\right)^{\xi_2},$$
$$x \in (0, \bar{x}_c(x^*))$$

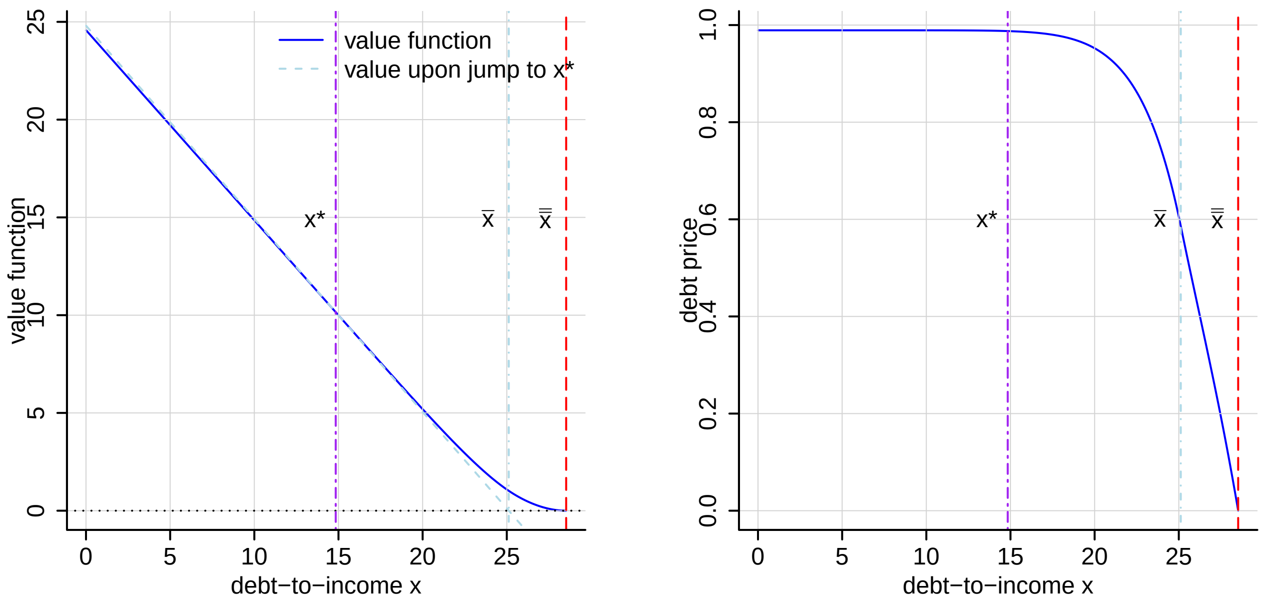
$$v_{\Delta,c}(x;x^*) = \frac{1}{\delta + 1/\Delta - \mu} - \frac{\kappa + m}{\delta + m + \Delta}x + v_1\left(\frac{x}{\bar{\bar{x}}_c(x^*)}\right)^{\xi_1} + v_2\left(\frac{x}{\bar{\bar{x}}_c(x^*)}\right)^{\xi_2} \qquad x \in (\bar{x}_c(x^*), \bar{\bar{x}}_c(x^*))$$

Given $x^*, \bar{x}_c(x^*)$ and $\bar{\bar{x}}_c(x^*)$, we have 4 unknown constants $(v^*, \bar{v}, v_1, v_2)$ to determine. The requirement that $v(x^*;x^*) = v^*$ delivers one equation. The requirement that v be $\mathcal{C}^1$ at $x = \bar{x}_c(x^*)$ delivers two more equations. The requirement that $v(\bar{\bar{x}}_c(x^*);x^*) = 0$ delivers one last equation. Note that this system of 4 equations in 4 unknown is a linear system.

For a given x^*, we are left with only $\bar{x}_c(x^*)$ and $\bar{\bar{x}}_c(x^*)$ to determine. These barriers solve a system

Figure 11: Value function and debt price with infrequent trades and commitment

(a): value function $v_{\Delta,c}$ **(b):** debt price $d_{\Delta,c}$



Plots show the value function and debt price for an expected trading interval $\Delta = 1$ year, in the case where the government can commit to always returning to a debt-to-income ratio x^*. Plots computed assuming $\mu = 2\%$ p.a., $\sigma = 20\%$ p.a., $1/m = 10$ years, $\theta = 0\%$, $\alpha = 0\%$, $\nu = 0\%$, $r = \kappa = 5\%$ and $\delta = 7\%$.

of 2 non-linear equations. The first non-linear equation comes from the fact that for $x = \bar{x}_c(x^*)$, the government who has an opportunity to change its debt-to-income ratio is exactly indifferent between (a) buying back its bonds to jump back to a debt-to-income ratio x^*, or (b) defaulting. In other words,

$$v_{\Delta,c}\left(x^*;x^*\right) + (x^* - \bar{x}_c(x^*))d_{\Delta,c}\left(x^*;x^*\right) = 0$$

Finally, default optimality yields the smooth pasting condition

$$v'_{\Delta,c}\left(\bar{\bar{x}}_c(x^*);x^*\right) = 0$$

These last two equations allow us to pin down $\bar{x}_c(x^*)$ and $\bar{\bar{x}}_c(x^*)$ given a choice of debt-to-income commitment x^*. At time zero, the government chooses the optimal debt-to-income x^* to commit to, in other words the government solves

$$\max_{x^*} v_{\Delta,c}(x^*;x^*) + x^* d_{\Delta,c}(x^*;x^*)$$

Figure 11 illustrates the value function and debt price for a particular choice of Δ. In the region $x < \bar{x}_c$, the debt-to-income ratio of the sovereign moves due to both Brownian income shocks and Poisson adjustment events. When the debt-to-income ratio is inside the interval $(\bar{x}_c, \bar{\bar{x}}_c)$, Poisson arrivals of trading opportunities introduce jump-to-default risk for creditors. At $x = \bar{x}_c$, upon the arrival of an adjustment opportunity, the government is exactly indifferent between defaulting and adjusting its debt-to-income ratio to x^*.

Finally, Figure 12 illustrates how the optimal default boundaries and optimal initial debt-to-income ratio change with the trading time interval Δ. As expected, the lower the expected trading interval, the closer the default boundaries $\bar{x}_c, \bar{\bar{x}}_c$ and x^* are from each other, and the closer they are to the first-best debt-to-income target leverage $1/(r-\mu)$. Similarly, the greater the trading time interval, the lower the optimal debt-to-income ratio x^*. □

Figure 12: Optimal boundaries in infrequent trading models

(a): Optimal default boundaries **(b):** Target debt-to-income ratios

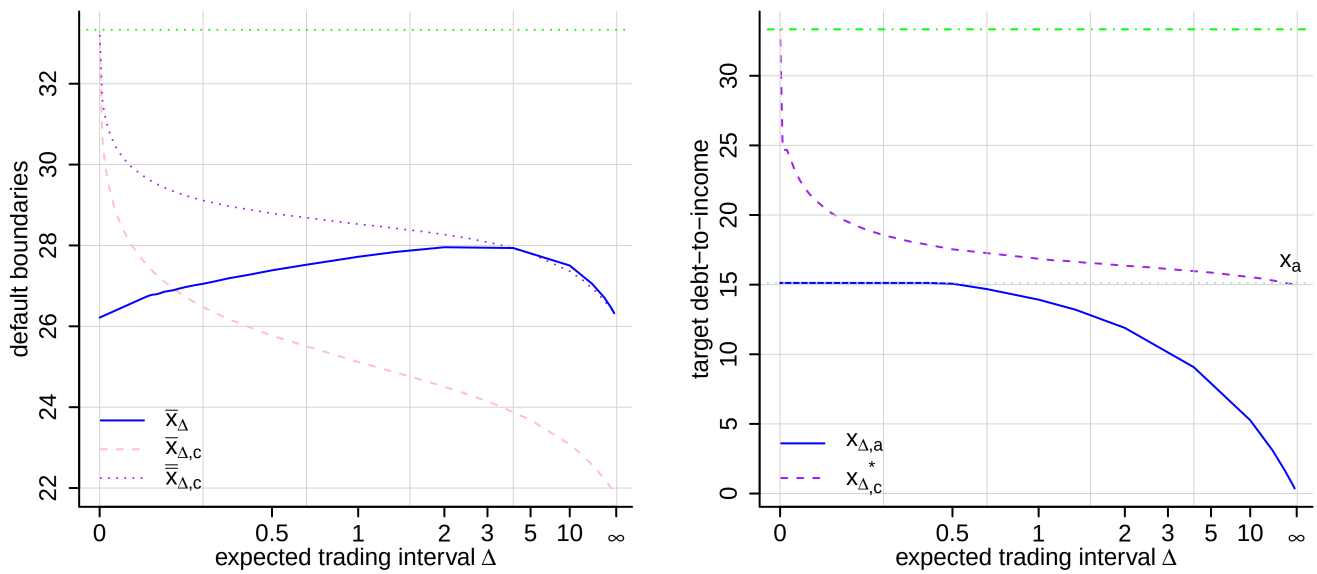


Figure (a) shows (i) the default boundary $\bar{x}_\Delta$ of the no-commitment model developed in Section C.1.1 (blue line) and (ii) the default boundaries $\bar{x}_{\Delta,c}$ and $\bar{\bar{x}}_{\Delta,c}$ of corresponding model with commitment and developed in Section C.1.2 (pink and purple lines). Figure (b) shows (i) the debt-to-income attraction point $x_{\Delta,a}$ of the no-commitment model (blue line, defined in equation (50)) and (ii) the optimal commitment debt-to-income x^* in the corresponding model with commitment (purple line). The dotted light blue line is the attraction point x_a in the continuous trading no-commitment model of Section B.1. The plot is computed assuming $\mu = 2\%$ p.a., $\sigma = 20\%$ p.a., $1/m = 10$ years, $\theta = 0\%$, $\alpha = 0\%$, $\nu = 0\%$, $r = \kappa = 5\%$ and $\delta = 7\%$.

C.2 Markov Switching Issuance Constraint

The government alternates between a constrained and unconstrained state at Poisson arrival times. When unconstrained (state "u"), the regime transitions to the constrained regime (state "c") with intensity λ_u. When constrained, the regime transitions to the unconstrained regime with intensity λ_c. We note v_u (resp. d_u) the value function for the government (resp. the debt price) when unconstrained, and v_c (resp. d_c) the value function for the government (resp. the debt price) when constrained.

We postulate that a Smooth MPE exists, in which, when in the unconstrained state "u," the government uses an absolutely continuous debt face value policy as in the Smooth MPE of Section 5 of the main text. This must then mean that the value functions in the constrained and unconstrained states v_c and v_u are equal to the value function in the no-commitment MPE v: $v_u(x) = v_c(x) = v(x)$ for all

$x \in [0, \bar{x}]$. In such case, we know the debt price in state "u" satisfies:

$$d_u(x) = -v_u'(x) = \frac{\kappa + m}{\delta + m}\left[1 - \left(\frac{x}{\bar{x}}\right)^{\xi - 1}\right].$$

In the constrained regime "c," the debt price d_c satisfies the following Feynman-Kac equation:

$$(r + m + \lambda_c)d_c(x) = \kappa + m - \left(m + \mu - \sigma^2\right) x d_c'(x) + \frac{\sigma^2 x^2}{2} d_c''(x) + \lambda_c d_u(x).$$

Recall $\eta > 0$ is the positive root of the quadratic equation $\frac{\sigma^2}{2}\eta^2 - \left(m + \mu - \frac{\sigma^2}{2}\right)\eta - (r + m + \lambda_c) = 0$, and recall that $\eta < \xi - 1$ if and only if $\delta > r + \lambda_c$. Using $d_c(\bar{x}) = 0$, we compute the debt price d_c as:

$$d_c(x) = \frac{\kappa + m}{r + m + \lambda_c}\left(1 + \frac{\lambda_c}{\delta + m}\right) + \frac{\lambda_c}{\delta - (r + \lambda_c)}\frac{\kappa + m}{\delta + m}\left(\frac{x}{\bar{x}}\right)^{\xi - 1}$$
$$- \left[\frac{\kappa + m}{r + m + \lambda_c}\left(1 + \frac{\lambda_c}{\delta + m}\right) + \frac{\lambda_c}{\delta - (r + \lambda_c)}\frac{\kappa + m}{\delta + m}\right]\left(\frac{x}{\bar{x}}\right)^{\eta}.$$

Note that irrespective of the parameter $\lambda_c > 0$, some tedious algebra allows us to show that $d_c'(x) < 0$; this guarantees that d_c is decreasing on $[0, \bar{x}]$.

We then combine the differential equation satisfied by v' with that satisfied by d_u:

$$(\delta + m)\, v'(x) = -(\kappa + m) - \left(m + \mu - \sigma^2\right) x v''(x) + \frac{\sigma^2 x^2}{2} v'''(x)$$
$$(r + m)\, d_u(x) = \kappa + m + \left[g_u(x) - \left(m + \mu - \sigma^2\right) x\right] d_u'(x) + \frac{\sigma^2 x^2}{2} d_u''(x) + \lambda_u \left(d_c(x) - d_u(x)\right).$$

Adding up those two equations, and using $d_u(x) + v'(x) = 0$ we obtain the issuance policy g_u in the unconstrained state "u":

$$g_u(x) = (\delta - r)\,\frac{d_u(x)}{-d_u'(x)} + \lambda_u \frac{d_c(x) - d_u(x)}{-d_u'(x)}.$$

This issuance policy is positive across the state space. Indeed, $\delta > r$, d_u is strictly decreasing in x and it is easy to verify that $\Delta d(x) := d_c(x) - d_u(x) > 0$. To see this, $\Delta d(x)$ satisfies the ODE:

$$(r + m + \lambda_c)\,\Delta d(x) = (\delta - r)d_u(x) - \left(m + \mu - \sigma^2\right) x \Delta d'(x) + \frac{\sigma^2}{2} x^2 \Delta d''(x). \tag{51}$$

The boundary condition $\Delta d(\bar{x}) = 0$ gives us an integral representation for $\Delta d(x)$:

$$\Delta d(x) = \mathbb{E}_x^{nt}\left[\int_0^{\tau} e^{-(r + m + \lambda_c)t}(\delta - r) d_u(x_t) dt\right] \geq 0,$$

where $\mathbb{E}^{nt}$ is the expectation operator under the no-trade policy.

Finally, it might be counterfactual to observe $\Delta d(x) > 0$, i.e., a higher debt price in the constrained state. We stress that this is partly due to the fact that we rule out the possibility of a rising risk premium in the constrained state. Suppose instead that $\nu_u < \nu_c$; that is to say, in the sudden stop constrained state

"c", credit market risk premia increase relative to the unconstrained state "u." Then, we have

$$(\delta + m)\, d_u(x) = \kappa + m - \left(m + \mu - \sigma^2\right) x d_u'(x) + \frac{\sigma^2 x^2}{2} d_u''(x)$$
$$(r + m)\, d_c(x) = \kappa + m - \left(m + \mu - \sigma^2 - \nu_c \sigma\right) x d_c'(x) + \frac{\sigma^2 x^2}{2} d_c''(x) + \lambda_c \left(d_u(x) - d_c(x)\right).$$

Subtracting one equation from the other, our previous equation (51) becomes

$$(r + m + \lambda_c)\, \Delta d(x) = (\delta - r) d_u(x) + \nu_c \sigma x d_c'(x) - \left(m + \mu - \sigma^2\right) x \Delta d'(x) + \frac{\sigma^2}{2} x^2 \Delta d''(x),$$

which means that its integral representation is now

$$\Delta d(x) = \mathbb{E}_x^{nt} \left[\int_0^\tau e^{-(r+m+\lambda_c)t} \left[(\delta - r) d_u(x_t) + \nu_c \sigma x d_c'(x) \right] dt \right].$$

Thus, if the risk premium ν_c is large enough, it is easy to obtain $\Delta d(x) < 0$ for *some* debt to income levels. Note finally that in such case, the financing policy becomes

$$g_u(x) = \left(\delta - (r + \pi_u(x))\right) \frac{d_u(x)}{-d_u'(x)} + \lambda_u \frac{d_c(x) - d_u(c)}{-d_u'(x)} \qquad\qquad \pi_u(x) := -\nu_u \sigma x d_u'(x).$$

□

C.3 Debt Ceiling Policies

In this section, we assume that there is a debt-to-income limit $x^* < \bar{x}$ such that if the small open economy's debt-to-income is above such threshold, the government is prevented from issuing any debt.

C.3.1 Smooth Equilibrium

We derive a condition on x^* such that our Smooth MPE still exists. When that is the case, the value function v_c is identical to the value function v in the unconstrained economy. The debt price d_c is such that when $x < x^*$, the debt price satisfies:

$$d_c(x) = \frac{\kappa + m}{\delta + m} \left[1 - \left(\frac{x}{\bar{x}} \right)^{\xi - 1} \right].$$

When $x \in (x^*, \bar{x})$, the debt price satisfies the following ODE:

$$(r + m) d_c(x) = \kappa + m - \left(m + \mu - \sigma^2\right) x d_c'(x) + \frac{\sigma^2 x^2}{2} d_c''(x),$$

with two boundary conditions at the points x^* and $\bar{x}$:

$$d_c(\bar{x}) = 0 \qquad\qquad d_c(x^*) = \frac{\kappa + m}{\delta + m} \left[1 - \left(\frac{x^*}{\bar{x}} \right)^{\xi - 1} \right].$$

Let $\eta_1 < 0 < \eta_2$ be the roots of the quadratic equation

$$\frac{\sigma^2}{2}\eta^2 - \left(m + \mu - \frac{\sigma^2}{2}\right)\eta - (r+m) = 0.$$

Since $\delta > r$, it is easy to verify that $\eta_2 < \xi - 1$. We can derive the debt price for $x \in (x^*, \bar{x})$ to be

$$d_c(x) = \frac{\kappa + m}{r+m} + d_1\left(\frac{x}{\bar{x}}\right)^{\eta_1} + d_2\left(\frac{x}{\bar{x}}\right)^{\eta_2},$$

where $\rho := x^*/\bar{x} \in (0,1)$ and

$$\begin{aligned} d_1 &= \frac{1}{\rho^{\eta_1} - \rho^{\eta_2}}\left[\frac{\kappa+m}{\delta+m}\left(1-\rho^{\xi-1}\right) - \frac{\kappa+m}{r+m}\left(1-\rho^{\eta_2}\right)\right] \\ d_2 &= \frac{1}{\rho^{\eta_1} - \rho^{\eta_2}}\left[-\frac{\kappa+m}{\delta+m}\left(1-\rho^{\xi-1}\right) + \frac{\kappa+m}{r+m}\left(1-\rho^{\eta_1}\right)\right]. \end{aligned}$$

In order for this to be an equilibrium, a necessary and sufficient condition is that the debt price d_c is decreasing on $[x^*, \bar{x}]$. For this to be the case, a sufficient condition is that it is decreasing at $x = x^*_+$, i.e.,

$$\eta_1 d_1 \rho^{\eta_1} + \eta_2 d_2 \rho^{\eta_2} = \frac{\rho^{\eta_1+\eta_2}}{\rho^{\eta_1} - \rho^{\eta_2}}\left[\underbrace{\left(\eta_2\rho^{-\eta_1} - \eta_1\rho^{-\eta_2}\right)\left(\frac{\kappa+m}{r+m} - \frac{\kappa+m}{\delta+m}(1-\rho^{\xi-1})\right) - \frac{\kappa+m}{r+m}(\eta_2-\eta_1)}_{:=F(\rho)}\right] < 0.$$

Let $F(\rho)$ be the function in brackets above. One can show that there is a unique ρ^* that satisfies $F(\rho^*) = 0$, with $F(\rho) > 0$ when $\rho < \rho^*$ and $F(\rho) < 0$ when $\rho > \rho^*$.[1] That is to say, our conjectured smooth equilibrium is indeed an equilibrium if and only if $x^* > \rho^*\bar{x} := \bar{x}^*$.

What remains to discuss is the fact that the debt price function d_c is continuous but not continuously differentiable at $x = x^*$. Suppose that d_c exhibits a convex kink:

$$\lim_{x \nearrow x^*} d_c'(x) < \lim_{x \searrow x^*} d_c'(x).$$

To rule out arbitrages, the government uses an issuance policy such that the controlled debt-to-income ratio becomes a skew Brownian motion (see (Harrison and Shepp, 1981)):

$$dx_t = \left[g(x_t) - \left(m + \mu - \sigma^2\right)x_t\right]dt - \sigma x_t dB_t + (2p-1)dL_t^{x^*}(x_t).$$

In the above, $L_t^{x^*}(x_t)$ is the local time at x^* of x_t:

$$L_t^{x^*}(x_t) := \lim_{\epsilon \searrow 0}\frac{1}{2\epsilon}\int_0^t 1_{\{x^*-\epsilon < x_u \le x^*+\epsilon\}}du.$$

[1]To see this, one can show that F is convex, with limit $+\infty$ as $\rho \to 0$ and limit 0 when $\rho \to 1$. It can also be shown that $F'(\rho) \to -\infty$ as $\rho \to 0$ and that $F'(\rho) \to \frac{\kappa+m}{\delta+m}(\eta_2-\eta_1)(\xi-1) > 0$ as $\rho \to 1$. The conclusion then follows.

The probability $p \in (0,1)$ of "moving to the right" is equal to:

$$p = \frac{\lim_{x \nearrow x^*} d'(x)}{\lim_{x \nearrow x^*} d'(x) + \lim_{x \searrow x^*} d'(x)}.$$

x_t is thus singular at x^* only, and one can think of the Skew Brownian motion as a way to distort probabilities of moving up or down at $x = x^*$, so that in expectations debt investors do not realize infinite (or minus infinite) capital gains' rates. □

C.3.2 Reflecting Equilibrium

Consider now $x^* < \bar{x}^*$ so that a smooth equilibrium does not exist. We conjecture that there is an equilibrium in which the debt-to-income ratio is evolving "unregulated" on $[x^*, \bar{x}_c]$, and is reflected at $x = x^*$ via singular control. In other words, x_t is now a regulated Brownian motion, regulated at $x = x^*$. The default boundary $\bar{x}_c$ is now different from the smooth equilibrium default boundary $\bar{x}$. The government value function and the debt price are then pinned down on $[x^*, \bar{x}_c]$, independently of what happens when $x < x^*$. For $x < x^*$, as we will see, two situations can arise.

1. Suppose that $d_c(x^*) < \frac{\kappa+m}{\delta+m}$ (in other words if x^* is sufficiently close to $\bar{x}^*$ and if δ is not "too large"), and $\hat{x} \in (0, x^*)$ (where $\hat{x}$ is defined in equation (52)). Then there exists a jump region $[\hat{x}, x^*]$, in which the government finds it optimal to jump immediately to x^*, and a "smooth" region $[0, \hat{x}]$, in which the government finds it optimal to follow a smooth debt issuance strategy.

2. If $d_c(x^*) > \frac{\kappa+m}{\delta+m}$, or if $\hat{x} \notin (0, x^*)$, then only the jump region exists.

In both cases, for $x \in (x^*, \bar{x}_c)$, the value function v_c and the debt price d_c satisfy the following:

$$(\delta - \mu) v_c(x) = 1 - (\kappa + m)x - (m + \mu)\, x v_c'(x) + \frac{\sigma^2 x^2}{2} v_c''(x)$$
$$(r + m) d_c(x) = \kappa + m - \left(m + \mu - \sigma^2\right) x d_c'(x) + \frac{\sigma^2 x^2}{2} d_c''(x),$$

with the boundary conditions:

$$v_c(\bar{x}_c) = 0 \qquad\qquad d_c(\bar{x}_c) = 0$$
$$v_c'(x^*) + d_c(x^*) = 0 \qquad\qquad d_c'(x^*) = 0.$$

The first two conditions are standard, as they correspond to value-matching conditions at $x = \bar{x}_c$. The last two conditions are standard boundary conditions for regulated Brownian motions. Let $\xi_1 < 0 < \xi_2$ be the roots of

$$\frac{\sigma^2}{2}\xi^2 - \left(m + \mu + \frac{\sigma^2}{2}\right)\xi - (\delta - \mu) = 0.$$

Let $\eta_1 < 0 < \eta_2$ be the roots of

$$\frac{\sigma^2}{2}\eta^2 - \left(m + \mu - \frac{\sigma^2}{2}\right)\eta - (r + m) = 0.$$

Since $\delta > r$, it is easy to verify that $\eta_2 < \xi_2 - 1$. Note $\rho := x^*/\bar{x}_c$. The debt price for $x \in (x^*, \bar{x}_c)$ thus satisfies:

$$d_c(x) = \frac{\kappa + m}{r + m} + d_1 \left(\frac{x}{\bar{x}_c}\right)^{\eta_1} + d_2 \left(\frac{x}{\bar{x}_c}\right)^{\eta_2}.$$

The constants of integration d_1, d_2 satisfy:

$$d_1 = \frac{\kappa + m}{r + m} \left(\frac{\eta_2 \rho^{\eta_2}}{\eta_1 \rho^{\eta_1} - \eta_2 \rho^{\eta_2}}\right) \qquad d_2 = \frac{\kappa + m}{r + m} \left(\frac{-\eta_1 \rho^{\eta_1}}{\eta_1 \rho^{\eta_1} - \eta_2 \rho^{\eta_2}}\right).$$

The value function v_c satisfies:

$$v_c(x) = \frac{1}{\delta - \mu} - \frac{\kappa + m}{\delta + m} x + v_1 \left(\frac{x}{\bar{x}_c}\right)^{\xi_1} + v_2 \left(\frac{x}{\bar{x}_c}\right)^{\xi_2}.$$

The constants of integration v_1, v_2 satisfy:

$$\begin{aligned} v_1 &= \frac{\bar{x}_c}{\xi_1 \rho^{\xi_1 - 1} - \xi_2 \rho^{\xi_2 - 1}} \left[\frac{\kappa + m}{\delta + m}\left(1 - \xi_2 \rho^{\xi_2 - 1}\right) + \xi_2 \rho^{\xi_2 - 1} \frac{1}{\bar{x}_c(\delta - \mu)} - \frac{\kappa + m}{r + m}\left(1 + \frac{(\eta_2 - \eta_1)\rho^{\eta_1 + \eta_2}}{\eta_1 \rho^{\eta_1} - \eta_2 \rho^{\eta_2}}\right)\right] \\ v_2 &= \frac{\bar{x}_c}{\xi_1 \rho^{\xi_1 - 1} - \xi_2 \rho^{\xi_2 - 1}} \left[\frac{\kappa + m}{\delta + m}\left(\xi_1 \rho^{\xi_1 - 1} - 1\right) - \xi_1 \rho^{\xi_1 - 1} \frac{1}{\bar{x}_c(\delta - \mu)} + \frac{\kappa + m}{r + m}\left(1 + \frac{(\eta_2 - \eta_1)\rho^{\eta_1 + \eta_2}}{\eta_1 \rho^{\eta_1} - \eta_2 \rho^{\eta_2}}\right)\right]. \end{aligned}$$

Finally, the default optimality condition $v_c'(\bar{x}_c)$ pins down $\bar{x}_c$:

$$-\frac{\kappa + m}{\delta + m} \bar{x}_c + v_1 \xi_1 + v_2 \xi_2 = 0.$$

Now we consider the case of $x < x^*$. Imagine first that $d_c(x^*) < \frac{\kappa + m}{\delta + m}$. This condition is equivalent to:

$$d_c(x^*) < \frac{\kappa + m}{\delta + m} \Leftrightarrow v_1 \xi_1 \rho^{\xi_1} + v_2 \xi_2 \rho^{\xi_2} > 0.$$

In such case, one can construct an equilibrium in which the government follows a smooth issuance strategy for $x \in (0, \hat{x})$, and a jump strategy for $x \in (\hat{x}, x^*)$, for some cutoff $\hat{x}$ endogenously determined. In the jump region $[\hat{x}, x^*]$, the debt price must be constant and the value function must be linear in x:

$$\begin{aligned} v_c(x) &= v_c(x^*) + (x^* - x) d_c(x^*) \\ d_c(x) &= d_c(x^*). \end{aligned}$$

On the interval $[0, \hat{x}]$, since we postulated that the government follows a smooth financing policy, the value function must satisfy:

$$(\delta - \mu) v_c(x) = 1 - (\kappa + m)x - (m + \mu)\, x v_c'(x) + \frac{\sigma^2 x^2}{2} v_c''(x),$$

with boundary conditions $v_c(0) = \frac{1}{\delta - \mu}$ and $\lim_{x \searrow \hat{x}} v_c(x) = \lim_{x \nearrow \hat{x}} v_c(x)$. We can solve for $v_c(x)$ as:

$$v_c(x) = \frac{1}{\delta - \mu}\left[1 - \left(\frac{x}{\hat{x}}\right)^{\xi_2}\right] - \frac{\kappa + m}{\delta + m} x \left[1 - \left(\frac{x}{\hat{x}}\right)^{\xi_2 - 1}\right] + v_c(\hat{x}) \left(\frac{x}{\hat{x}}\right)^{\xi_2}.$$

Since $d_c(x) = -v_c'(x)$, we obtain the following expression for d_c:

$$d_c(x) = \frac{\kappa+m}{\delta+m} - \frac{\xi_2}{\hat{x}}\left[v_c(\hat{x}) + \frac{\kappa+m}{\delta+m}\hat{x} - \frac{1}{\delta-\mu}\right]\left(\frac{x}{\hat{x}}\right)^{\xi_2-1}.$$

The threshold $\hat{x}$ is pinned down by the continuity of d_c at $\hat{x}$. Since $d_c(\hat{x}-) + v_c'(\hat{x}-) = 0$ (due to the fact that the strategy is smooth on $[0,\hat{x}]$), and since $d_c(\hat{x}+) + v_c'(\hat{x}+) = 0$ (due to the fact that the value function is linear, with slope $-d_c(\hat{x}+)$, on $[\hat{x}, x^*]$), the requirement that d_c is continuous at $\hat{x}$ is identical to the requirement that v_c is $\mathcal{C}^1$ at such point. This condition can be shown to lead to:

$$\hat{x} = \frac{\xi_2}{1-\xi_2}\left(\frac{(1-\xi_1)\rho^{\xi_1}v_1 + (1-\xi_2)\rho^{\xi_2}v_2}{\xi_1\rho^{\xi_1}v_1 + \xi_2\rho^{\xi_2}v_2}\right)x^*. \tag{52}$$

If $\hat{x}$ in equation (52) satisfies $0 < \hat{x} < x^*$, then an MPE exists, in which (a) the financing policy of the government is smooth on $(0,\hat{x})$, (b) the government jumps to x^* if $x \in (\hat{x}, x^*)$, and (c) the debt-to-income ratio evolves with income shocks for $x > x^*$ and is reflected at $x = x^*$.

If instead (a) the constant $\hat{x}$ defined in equation (52) is outside the interval $[0, x^*]$, or (b) $d_c(x^*) > \frac{\kappa+m}{\delta+m}$, then the "smooth" region no longer exists, and one can construct an equilibrium in which the government follows a jump strategy for $x \in [0, x^*]$. On such interval, the debt price must be constant and the value function must be linear in x:

$$v_c(x) = v_c(x^*) + (x^* - x)d_c(x^*), \qquad d_c(x) = d_c(x^*)$$

□

C.4 Issuance Rate Cap

In this section, the issuance rate (per unit of income) is capped at some arbitrary constant $\bar{g} > 0$. We look for an equilibrium where the constraint binds whenever the debt-to-income ratio is below an endogenously determined cutoff x^*. For $x \in (x^*, \bar{x}_c)$, the constraint is slack, where $\bar{x}_c$ is the optimal default boundary. We take $x^*, \bar{x}_c$ as given in the analysis below, and then discuss the two conditions that pin down both endogenous boundaries.

C.4.1 Constrained Region $[0, x^*]$

In the region $x \in [0, x^*)$, the issuance rate is bounded above by $\bar{g}$. On this interval, the scaled welfare for the government and the debt price satisfy:

$$(\delta-\mu)\,v(x) = 1 + \bar{g}d(x) - (\kappa+m)x + [\bar{g} - (\mu+m)\,x]\,v'(x) + \frac{1}{2}\sigma^2x^2v''(x) \tag{53}$$

$$(r+m)\,d(x) = (\kappa+m) + \left[\bar{g} - \left(\mu+m-\sigma^2\right)x\right]d'(x) + \frac{1}{2}\sigma^2x^2d''(x) \tag{54}$$

Note that these ordinary differential equations are decoupled – we can solve for $d(\cdot)$ first, and reinject d into the ODE that v is solution of. The boundary conditions are as follows:

$$(\delta - \mu)v(0) = 1 + \bar{g}\left(d(0) + v'(0)\right) \qquad \lim_{x \nearrow x^*} v(x) = \lim_{x \searrow x^*} v(x)$$

$$(r+m)d(0) = \kappa + m + \bar{g}d'(0) \qquad \lim_{x \nearrow x^*} d(x) = \lim_{x \searrow x^*} d(x)$$

The boundary conditions at $x = 0$ are standard Robin boundary conditions, linking the value of the function to its derivative at that point. In what follow, we are going to treat $d(x^*)$ and $v(x^*)$ as parameters, and will eventually obtain equations that will tie $d(x^*)$ and $v(x^*)$ to the boundaries $x^*, \bar{x}_c$. We first establish the following lemma.

Lemma 4 *Let $A, B, C \in \mathbb{R}^3_+$. Let $f(x; A, B, C)$ be a $\mathcal{C}^2$ function defined on $[0; x^*]$ (and thus finite on that interval) that satisfies the second order ordinary differential equation:*

$$x^2 f''(x) + (A - Bx)f'(x) - Cf(x) = 0 \tag{55}$$

Then f takes the following form, for some coefficients $k_1, k_2 \in \mathbb{R}$:

$$f(x; A, B, C) = k_1 x^{-\eta_1} U\left(\eta_1; 2\eta_1 + B + 2; Ax^{-1}\right) + k_2 x^{-\eta_2} U\left(\eta_2; 2\eta_2 + B + 2; Ax^{-1}\right)$$

In the above, U is the Tricommi confluent hypergeometric function (see Abramowitz and Stegun (1964), chapter 13) and the constants $\eta_1 > 0 > \eta_2$ are the roots of the polynomial:

$$\eta^2 + (B+1)\eta - C = 0$$

The proof of the above lemma is straight-forward once we remember that Kummer's confluent hypergeometric function $M(a;b;z)$ and Tricommi's confluent hypergeometric function $U(a;b;z)$ are independent solutions to the Kummer differential equation:

$$zu''(z) + (b - z)\,u'(z) - au(z) = 0$$

It is then easy to check that $x^{-\eta} M\left(\eta; 2\eta + B + 2; Ax^{-1}\right)$ and $x^{-\eta} U\left(\eta; 2\eta + B + 2; Ax^{-1}\right)$ are solutions of equation (55). Note that M admits the asymptotic behavior $M(a;b;z) \sim e^z z^{a-b}/\Gamma(a)$ as $z \to +\infty$ and U admits the asymptotic behavior $U(a;b;z) \sim z^{-a}$ as $z \to +\infty$. In particular, f finite at $x = 0$ allows us to rule out the Kummer function and work with the Tricommi function only. □

Denote $\eta_{d,1} < 0 < 1 < \eta_{d,2}$ to be the roots of:

$$\frac{1}{2}\sigma^2 \eta_d^2 + \left(m + \mu - \frac{1}{2}\sigma^2\right)\eta_d - (r+m) = 0$$

We can use the previous lemma to show that:

$$d(x) = \frac{\kappa + m}{r + m} + k_{d,1} x^{-\eta_{d,1}} U\left(\eta_{d,1}; 2\eta_{d,1} + \frac{2(m+\mu)}{\sigma^2}; \frac{2\bar{g}}{\sigma^2 x}\right) + k_{d,2} x^{-\eta_{d,2}} U\left(\eta_{d,2}; 2\eta_{d,2} + \frac{2(m+\mu)}{\sigma^2}; \frac{2\bar{g}}{\sigma^2 x}\right).$$

The boundary conditions at $x = 0$ and $x = x^*$ then allow us to pin down $k_{d,1}, k_{d,2}$ uniquely as a functions of the (yet unknown) value $d(x^*)$. Then, given the function d fully specified on $[0, x^*]$, equation (53) is a second order boundary value problem, and Baxley and Brown (1981) provides for the existence and uniqueness of a solution to this ordinary differential equation. Note $\eta_{v,1} < 0 < 1 < \eta_{v,2}$ the roots of:

$$\frac{1}{2}\sigma^2 \eta_v^2 + \left(m + \mu + \frac{1}{2}\sigma^2\right)\eta_v - (\delta - \mu) = 0.$$

The function v takes the following form:

$$v(x) = \frac{1}{\delta - \mu}\left(1 - \frac{\kappa + m}{\delta + m}\bar{g}\right) - \frac{\kappa + m}{\delta + m}x + v_p(x) + k_{v,1} v_{g,1}(x) + k_{v,2} v_{g,2}(x).$$

In the above, the general solutions $v_{g,i}$ take the following form:

$$v_{g,i}(x) := v_i x^{-\eta_{v,i}} U\left(\eta_{v,i}; 2\eta_{v,i} + \frac{2(m+\mu)}{\sigma^2} + 2; \frac{2\bar{g}}{\sigma^2 x}\right),$$

and v_p is a particular solution to the ordinary differential equation:

$$(\delta - \mu)\, v(x) = \bar{g} d(x) + \left[\bar{g} - (\mu + m)\, x\right] v'(x) + \frac{1}{2}\sigma^2 x^2 v''(x).$$

One can show that $v_p(x) = v_{g,1}(x) u(x)$, with the function $u(x)$ satisfying:

$$H(x) := \exp\left[\int_{x^*}^{x} \frac{(\bar{g} - (m+\mu)s)\, v_{g,1}(s) + \sigma^2 s^2 v'_{g,1}(s)}{\frac{\sigma^2 s^2}{2} v_{g,1}(s)} ds\right]$$
$$u(x) := \int_{x^*}^{x}\left(\int_{x^*}^{t} \frac{-2\bar{g}}{\sigma^2 s^2 v_{g,1}(s)} \frac{H(s)}{H(t)} ds\right) dt.$$

Those solutions d and v are strictly decreasing on the interval $(0, x^*)$, under the assumption – to be verified numerically – that $d(x^*) < d(0)$ and $d'(0) < 0$ (for d) and under the assumption that $v(x^*) < v(0)$ and $v'(0) \leq 0$ for v. Indeed, assume for example by way of contradiction that d was not strictly decreasing on that interval. This means that there exists $0 < x_1 < x_2 < x^*$, such that $d(x_1) < d(x_2)$, $d'(x_1) = d'(x_2) = 0$, and $d''(x_1) > 0 > d'(x_2)$. But in that case, using equation (54), we have:

$$\frac{1}{2}\sigma^2 x_1^2 d''(x_1) = (r + m) d(x_1) - (\kappa + m) > 0$$
$$\frac{1}{2}\sigma^2 x_2^2 d''(x_2) = (r + m) d(x_2) - (\kappa + m) < 0.$$

In other words, $d(x_1) > d(x_2)$, a contradiction. A similar proof holds for v. We thus have determined v and d on the interval $[0, x^*]$, subject to our knowledge of $x^*, d(x^*), v(x^*)$.

We can then verify that the issuance constraint is binding – in other words, that if the government was allowed to issue debt at an intensity greater than $\bar{g}$, it would find it optimal to do so – this is identical to verifying that $d(x) + v'(x) \geq 0$. The unconstrained issuance policy $g_u(x)$ verifies $g_u(x) := \frac{d(x)}{-d'(x)}(\delta - r)$, and since in $(0, x^*)$ the government is constrained to issue an amount $\bar{g}$, we must have in this particular part of the state space $\bar{g} < g_u(x)$. Differentiate equation (53) to obtain:

$$(\delta + m)\, v'(x) = \bar{g} d'(x) - (\kappa + m) + \left[\bar{g} - \left(\mu + m - \sigma^2\right) x\right] v''(x) + \frac{1}{2}\sigma^2 x^2 v'''(x)$$
$$(r + m)\, d(x) = (\kappa + m) + \left[\bar{g} - \left(\mu + m - \sigma^2\right) x\right] d'(x) + \frac{1}{2}\sigma^2 x^2 d''(x).$$

Add those last two equations, introduce $h(x) := d(x) + v'(x)$, and note that h satisfies:

$$(\delta + m)\, h(x) = \left[\bar{g} - g_u(x)\right] d'(x) + \left[\bar{g} - \left(\mu + m - \sigma^2\right) x\right] h'(x) + \frac{1}{2}\sigma^2 x^2 h''(x).$$

Then use the boundary condition $(\delta - \mu)v(0) = 1 + \bar{g}\left(d(0) + v'(0)\right)$, and remember that it must be the case that $v(0) \geq \frac{1}{\delta - \mu}$ (in other words, the welfare of a government that has no debt, but that has the option to borrow from more patient lenders must be at least as high as the autarky welfare) to conclude that $d(0) + v'(0) \geq 0$, in other words $h(0) \geq 0$. At $x = x^*$, v must be $\mathcal{C}^1$ and d is continuous, meaning that we must have $d(x^*) + v'(x^*) = 0$, in other words $h(x^*) = 0$. Using Feynman-Kac, $h(x)$ thus admits the integral representation:

$$h(x) = \mathbb{E}_x \left[\int_0^{\tau_{x^*}} e^{-(\delta + m)t} \left(\bar{g} - g_u(x_t)\right) d'(x_t) dt \right]$$

The stopping time τ_{x^*} is the first time the state variable x hits x^*. Since $g_u(x) \geq \bar{g}$ in that region of the state space, since d is a decreasing functions of x, it must be the case that $h(x) \geq 0$. □

C.4.2 Unconstrained Region $[x^*, \bar{x}]$

Given our postulated behavior, in $x \in (x^*, \bar{x})$ the government financing policy is entirely unconstrained, meaning that the analysis we discussed in Section (5) of the main text is unchanged: the value function for the government behaves locally as if the government was committing not to issue any debt. Thus, the scaled welfare for the government, the debt price and the issuance policy satisfy:

$$(\delta - \mu)\, v(x) = 1 - (\kappa + m)\, x - (\mu + m)\, x v'(x) + \frac{1}{2}\sigma^2 x^2 v''(x)$$
$$d(x) = -v'(x)$$
$$g(x) = \frac{d(x)}{-d'(x)}(\delta - r).$$

These equations are derived using steps identical to those used in Section B.1. The debt price function is thus entirely pinned down by the equation $d(x) = -v'(x)$, and it can be showed that it satisfies the second order ordinary differential equation:

$$(\delta + m)\, d(x) = \kappa + m - \left(m + \mu - \sigma^2\right) x d'(x) + \frac{1}{2}\sigma^2 x^2 d''(x). \tag{56}$$

As discussed previously, equation (56) is the Feynman-Kac representation of the debt price computed using discount rate δ and under the assumption that the government never issues any additional bonds. Boundary conditions are as follows:

$$
\begin{aligned}
v(\bar{x}) &= 0 \qquad & \lim_{x \nearrow x^*} v(x) &= \lim_{x \searrow x^*} v(x) \\
d(\bar{x}) &= 0 \qquad & \lim_{x \nearrow x^*} d(x) &= \lim_{x \searrow x^*} d(x).
\end{aligned}
$$

The government value function, debt price and issuance policy take the following form on $x \in [x^*, \bar{x}]$:

$$
\begin{aligned}
v(x) &= \frac{1}{\delta - \mu} - \left(\frac{\kappa + m}{\delta + m}\right) x + v_1 \left(\frac{x}{\bar{x}}\right)^{\xi_1} + v_2 \left(\frac{x}{\bar{x}}\right)^{\xi_2} \\
d(x) &= \frac{\kappa + m}{\delta + m} + d_1 \left(\frac{x}{\bar{x}}\right)^{\xi_1 - 1} + d_2 \left(\frac{x}{\bar{x}}\right)^{\xi_2 - 1} \\
g(x) &= (r - \delta) x \frac{\frac{\kappa + m}{\delta + m} + d_1 \left(\frac{x}{\bar{x}}\right)^{\xi_1 - 1} + d_2 \left(\frac{x}{\bar{x}}\right)^{\xi_2 - 1}}{(\xi_1 - 1) d_1 \left(\frac{x}{\bar{x}}\right)^{\xi_1 - 1} + (\xi_2 - 1) d_2 \left(\frac{x}{\bar{x}}\right)^{\xi_2 - 1}}.
\end{aligned}
$$

Since $-v'(x) = d(x)$, the constants v_1, v_2, d_1, d_2 are linked via $d_i = -\xi_i v_i / \bar{x}$. $\xi_1 < 0 < 1 < \xi_2$ are the roots of the polynomial:

$$
\frac{1}{2}\sigma^2 \xi^2 - \left(\mu + m + \frac{1}{2}\sigma^2\right) \xi - (\delta - \mu) = 0.
$$

The boundary conditions for d and for v at $x = \bar{x}$ lead to:

$$
\begin{aligned}
d_1 + d_2 + \frac{\kappa + m}{\delta + m} &= 0 \\
v_1 + v_2 + \frac{1}{\delta - \mu} - \frac{\kappa + m}{\delta + m} \bar{x} &= 0.
\end{aligned}
$$

The boundary conditions for d and for v at $x = x^*$ lead to:

$$
\begin{aligned}
\frac{\kappa + m}{\delta + m} + d_1 \left(\frac{x^*}{\bar{x}}\right)^{\xi_1 - 1} + d_2 \left(\frac{x^*}{\bar{x}}\right)^{\xi_2 - 1} &= d(x^*) \\
\frac{1}{\delta - \mu} - \left(\frac{\kappa + m}{\delta + m}\right) x^* + v_1 \left(\frac{x^*}{\bar{x}}\right)^{\xi_1} + v_2 \left(\frac{x^*}{\bar{x}}\right)^{\xi_2} &= v(x^*).
\end{aligned}
$$

Note that the boundary condition for d at $x = \bar{x}$ is identical to the smooth-pasting default optimality condition at such point (this latter condition is thus redundant). At the boundary $x = x^*$, the debt issuance rate of the small open economy is equal to $\bar{g}$. This gives us the following equation:

$$
\bar{g} = (r - \delta) x \frac{\frac{\kappa + m}{\delta + m} + d_1 \left(\frac{x^*}{\bar{x}}\right)^{\xi_1 - 1} + d_2 \left(\frac{x^*}{\bar{x}}\right)^{\xi_2 - 1}}{(\xi_1 - 1) d_1 \left(\frac{x^*}{\bar{x}}\right)^{\xi_1 - 1} + (\xi_2 - 1) d_2 \left(\frac{x^*}{\bar{x}}\right)^{\xi_2 - 1}}.
$$

We need to make sure our initial choices $x^*, \bar{x}$ are such that $d(x^*) < d(0)$, which insures that the function d is monotone decreasing on $[0, x^*]$. □

C.4.3 Determination of x^* and $\bar{x}$

It remains to discuss how the boundaries x^*, $\bar{x}$ are optimally set by the government. To be able to apply a standard verification theorem, we need to smooth the value function v, in other words, x^*, $\bar{x}$ are determined via the two smooth pasting conditions:

$$\lim_{x \nearrow x^*} v'(x; x^*, \bar{x}) = \lim_{x \searrow x^*} v'(x; x^*, \bar{x})$$
$$\lim_{x \searrow \bar{x}} v'(x; x^*, \bar{x}) = 0.$$

Assuming that there exists a solution to this two-equation, two-unknown system, we then have our main result: when the government is constrained to use an issuance rate below a certain maximum level $\bar{g}$, an equilibrium exists, in which the issuance policy is unconstrained for $x > x^*$, and constrained at $\bar{g}$ when $x \in (0, x^*)$. It is optimal for the government to default as soon as x reaches $\bar{x}$. In that equilibrium the welfare of a government that is not indebted is strictly greater than the autarky welfare. □

C.4.4 Simplification: Case $\sigma = 0, \mu + m < 0$

In this particular case, we can solve for v and d in closed form. We have $\bar{x} = 1/(\kappa + m)$. When $x < x^*$, the debt price and government value function take the following expressions:

$$d(x) = \frac{\kappa + m}{r + m} - \left(\frac{\kappa + m}{r + m} - d(x^*)\right)\left(\frac{\bar{g} - (\mu + m)x}{\bar{g} - (\mu + m)x^*}\right)^{-\frac{r+m}{\mu+m}}$$
$$v(x) = a_0 + a_1 x + a_2 \left(\frac{\bar{g} - (\mu + m)x}{\bar{g} - (\mu + m)x^*}\right)^{-\frac{r+m}{\mu+m}} + (v(x^*) - a_0 - a_1 x^* - a_2)\left(\frac{\bar{g} - (\mu + m)x}{\bar{g} - (\mu + m)x^*}\right)^{-\frac{\delta-\mu}{\mu+m}}.$$

In the above, the constants a_0, a_1, a_2 are equal to:

$$a_0 := \frac{1}{\delta - \mu}\left[1 + \frac{\bar{g}(\kappa + m)(\delta - r)}{(r + m)(\delta + m)}\right]$$
$$a_1 := -\frac{\kappa + m}{\delta + m}$$
$$a_2 := \frac{\bar{g}}{\delta - \mu - r - m}\left(d(x^*) - \frac{\kappa + m}{r + m}\right).$$

When $x \in (x^*, \bar{x})$, the debt price, government value function and issuance policy take the following expressions:

$$d(x) = \frac{\kappa + m}{r + m}\left[1 - \left(\frac{x}{\bar{x}}\right)^{-\frac{r+m}{\mu+m}}\right]$$
$$v(x) = \frac{1}{\delta - \mu} - \frac{\kappa + m}{\delta + m}x - \left[\frac{1}{\delta - \mu} - \frac{\kappa + m}{\delta + m}\bar{x}\right]\left(\frac{x}{\bar{x}}\right)^{-\frac{\delta-\mu}{\mu+m}}$$
$$g(x) = -\frac{(\delta - \mu)(\mu + m)}{\delta + m}\left[\left(\frac{\bar{x}}{x}\right)^{-\frac{\delta+m}{\mu+m}} - 1\right]x.$$

Since $g(x^*) = \bar{g}$, x^* is determined as the solution of the implicit equation:

$$\bar{g} = -\frac{(\delta-\mu)(\mu+m)}{\delta+m}\left[\left(\frac{\bar{x}}{x^*}\right)^{-\frac{\delta+m}{\mu+m}} - 1\right] x^*$$

C.5 Risk Aversion

C.5.1 General Treatment

We assume that the income Y_t is a (μ, σ) geometric Brownian motion. The government solves

$$V_t := \sup_{(G,\tau)} \mathbb{E}\left[\int_t^{+\infty} e^{-\delta(s-t)} \frac{C_s^{1-\gamma}}{1-\gamma} ds\right]$$

The resource constraint is $C_t = Y_t + G_t D_t - (\kappa+m)F_t$. At a default time τ, the country suffers a downward income jump such that $Y_\tau = \alpha Y_{\tau-}$, and the debt-to-income ratio of the country is renegotiated, so that $F_\tau / Y_\tau = \theta F_{\tau-}/Y_{\tau-}$. In particular, this means that debt holders see their claim haircut by a factor $\alpha\theta$ upon a sovereign default. Without financial contract, the autarky government welfare V_{aut} is equal to

$$V_{aut}(Y_t) = \frac{Y_t^{1-\gamma}}{\rho(1-\gamma)} := Y_t^{1-\gamma} v_{aut}, \qquad \rho := \delta - (1-\gamma)\left(\mu - \frac{\gamma\sigma^2}{2}\right)$$

Going forward, we thus assume that the parameters $(\mu, \sigma, \gamma, \delta)$ jointly satisfy $\rho > 0$, and that the parameters $(\mu, \sigma, \gamma, \delta, \alpha, \theta)$ are such that equilibrium consumption when $\gamma = 0$ is always strictly positive. Let us introduce the probability measure $\tilde{\Pr}$, that satisfies

$$\tilde{\Pr}(A) = \mathbb{E}\left[e^{-\frac{(1-\gamma)^2\sigma^2}{2}t + (1-\gamma)\sigma B_t} 1_A\right].$$

Using this probability measure, one can show that the life-time utility for the government satisfies

$$\begin{aligned}
V_t &= \sup_{(G,\tau)} \mathbb{E}_t\left[\int_t^{+\infty} e^{-\delta(s-t)} \frac{(Y_s + G_s D_s - (\kappa+m)F_s)^{1-\gamma}}{1-\gamma} ds\right] \\
&= Y_t^{1-\gamma} \sup_{(g,\tau)} \mathbb{E}_t\left[\int_t^{+\infty} e^{-\delta(s-t)} \left(\frac{Y_s}{Y_t}\right)^{1-\gamma} \frac{(1 + g_s D_s - (\kappa+m)x_s)^{1-\gamma}}{1-\gamma} ds\right] \\
&= Y_t^{1-\gamma} \sup_{(g,\tau)} \tilde{\mathbb{E}}_t\left[\int_t^{+\infty} e^{-a(s-t)} \alpha^{(1-\gamma)(N_s - N_t)} \frac{(1 + g_s D_s - (\kappa+m)x_s)^{1-\gamma}}{1-\gamma} ds\right]
\end{aligned}$$

In the above, N_t is the counting process for default events. Note that the debt-to-income ratio, under the measures Pr and $\tilde{\Pr}$, satisfies

$$\begin{aligned}
dx_t &= \left[g(x_t) - \left(m + \mu - \sigma^2\right) x_t\right] dt - \sigma x_t dB_t - x_{t-}(1-\theta) dN_t \\
&= \left[g(x_t) - \left(m + \mu - \gamma\sigma^2\right) x_t\right] dt - \sigma x_t d\tilde{B}_t - x_{t-}(1-\theta) dN_t
\end{aligned}$$

Finally, the debt price satisfies

$$D_t := \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{+\infty} e^{-(r+m)(s-t)}\left(\alpha\theta\right)^{N_s-N_t}(\kappa+m)ds\right]$$

In that case, note that $D(Y,F) := d(x)$ and $V(Y,F) := Y^{1-\gamma}v(x)$. The HJB equation for the value function and the Feynman-Kac equation for debt prices take the following form:

$$\rho v(x) = \max_g \frac{(1+gd(x)-(\kappa+m)x)^{1-\gamma}}{1-\gamma} + \left[g-(\mu+m-\gamma\sigma^2)x\right]v'(x) + \frac{1}{2}\sigma^2x^2v''(x)$$
$$(r+m)d(x) = \kappa+m+\left[g(x)-\left(\mu+m-\sigma^2-\nu\sigma\right)x\right]d'(x)+\frac{1}{2}\sigma^2x^2d''(x).$$

The first order condition w.r.t. the financing policy g leads to:

$$d(x)c(x)^{-\gamma}+v'(x)=0,$$

where $c(x) := 1+g(x)d(x)-(\kappa+m)x$. This leads to the following issuance policy:

$$g(x) = \frac{1}{d(x)}\left[\left(\frac{-v'(x)}{d(x)}\right)^{-1/\gamma}+(\kappa+m)x-1\right].$$

The (value-matching) boundary conditions at default are as follows:

$$v(\bar{x}) = \alpha^{1-\gamma}v(\theta\bar{x}) \qquad\qquad d(\bar{x}) = \alpha\theta d(\theta\bar{x})$$

Default optimality takes the form of a smooth pasting condition $v'(\bar{x}) = \alpha^{1-\gamma}\theta v'(\theta\bar{x})$. Note that we can use the boundary conditions at $\bar{x}$ to also conclude that $c(\bar{x}) = \alpha c(\theta\bar{x})$ – in other words, the consumption path C_t must be continuous, even at default.

C.5.2 Numerical Solution for $\gamma > 0$

We numerically solve the MPE for the case of risk-averse government, with a risk-aversion coefficient $\gamma = 2$. Figure 13 shows the consumption policy c and issuance policy g, while Figure 14 shows the equilibrium prices d and credit spreads ς. In all these plots, we also show the ergodic distribution f.

C.5.3 Ergodic Moments

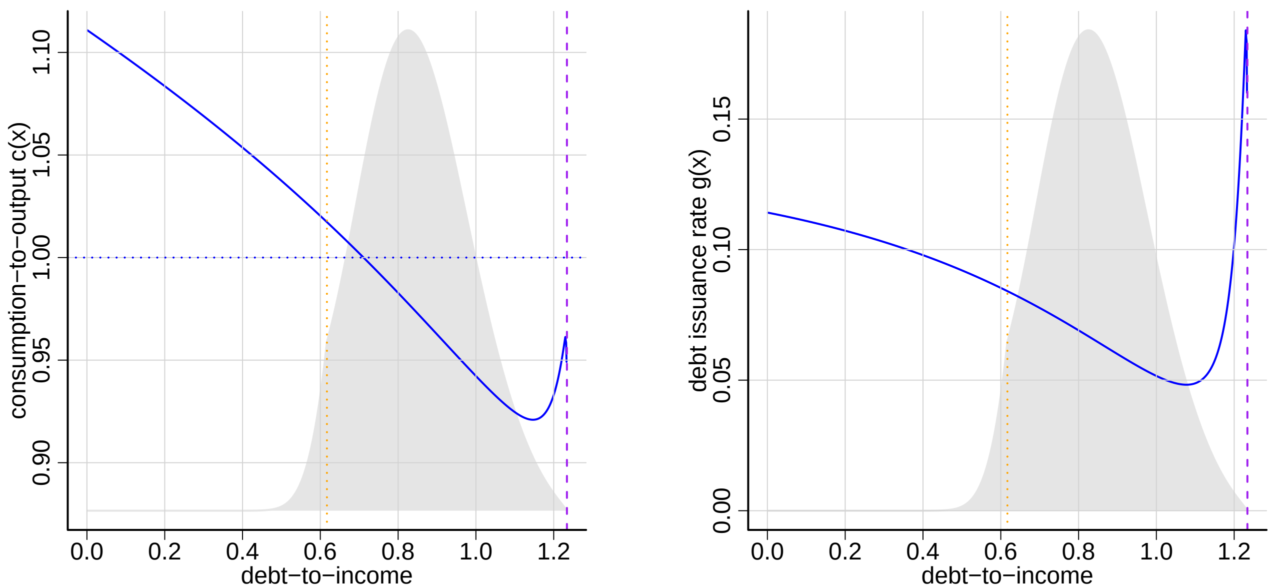
Armed with our computation of the ergodic density f_γ, we can also compute ergodic moments of interest. The left panel in Figure 15 plots the the stationary density f_γ and the default boundary $\bar{x}_\gamma$ for a range of parameters $\gamma \geq 0$. We also plot the ergodic average debt-to-income ratio $\mathbb{E}[x_t] = \int_0^{\bar{x}_\gamma} x f_\gamma(x)dx$ and the ergodic default frequency $\lambda_{d,\gamma} = -\frac{1}{2}\bar{x}_g^2 f'_\gamma(\bar{x}_g)$ in Figure 16.

C.5.4 Sensitivity to Debt Average Life $1/m$

Finally, when $\gamma = 2$, we study the sequence of equilibria indexed by the average debt maturity $1/m$. See Figure 17 and Figure 18.

Figure 13: Consumption and financing policies

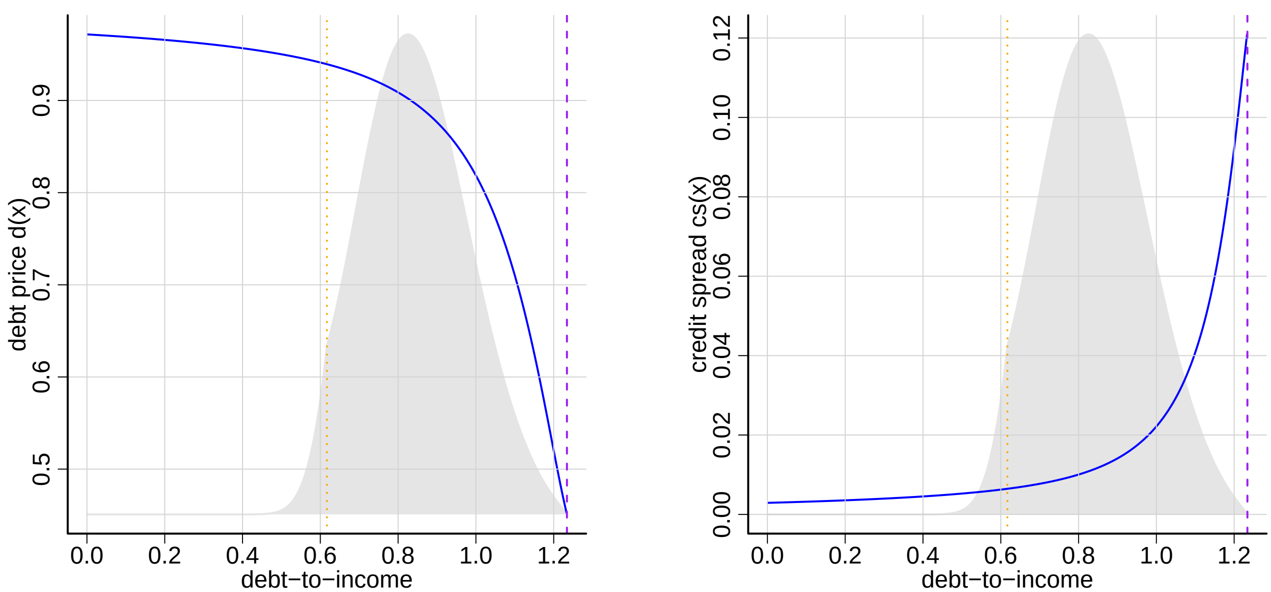
(a): Consumption-Output Ratio $c(x)$ **(b):** Debt Issuance $g(x)$



Plot computed assuming $r = \kappa = 5\%$, $\delta = 7\%$, $\mu = 2\%$, $\sigma = 10\%$, $1/m = 20$ years, $\theta = 50\%$, $1 - \alpha = 4\%$, $\gamma = 2$.

Figure 14: Debt prices and credit spreads

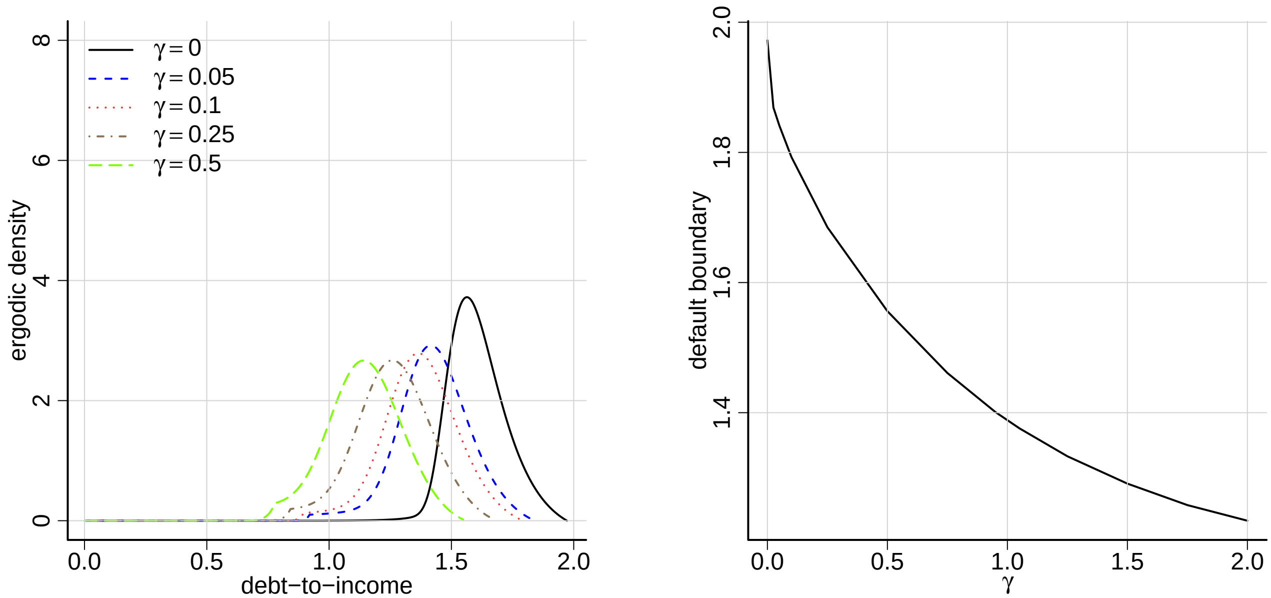
(a): Debt price $d(x)$ **(b):** Credit spread $cs(x)$



Plot computed assuming $r = \kappa = 5\%$, $\delta = 7\%$, $\mu = 2\%$, $\sigma = 10\%$, $1/m = 20$ years, $\theta = 50\%$, $1 - \alpha = 4\%$, $\gamma = 2$.

Figure 15: Ergodic density and default cutoff

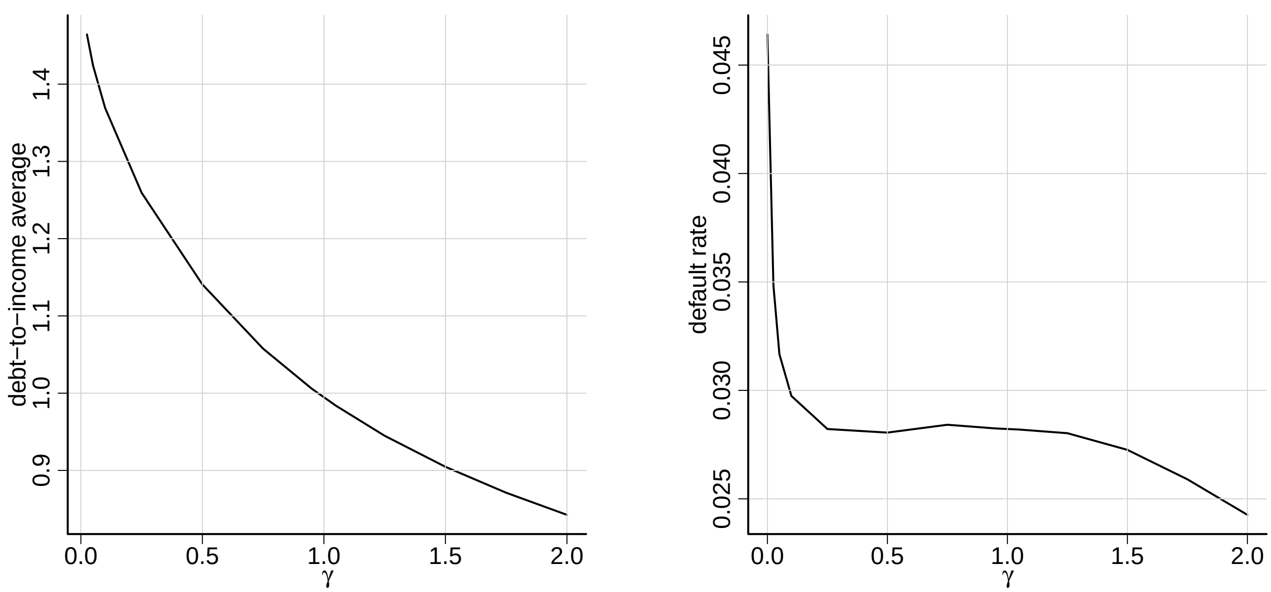
(a): ergodic density $f_\gamma(x)$ vs. γ | **(b):** Default boundary $\bar{x}_\gamma$ vs. γ



Plot computed assuming $r = \kappa = 5\%$, $\delta = 7\%$, $\mu = 2\%$, $\sigma = 10\%$, $1/m = 20$ years, $\theta = 50\%$, $1 - \alpha = 4\%$.

Figure 16: Ergodic moments

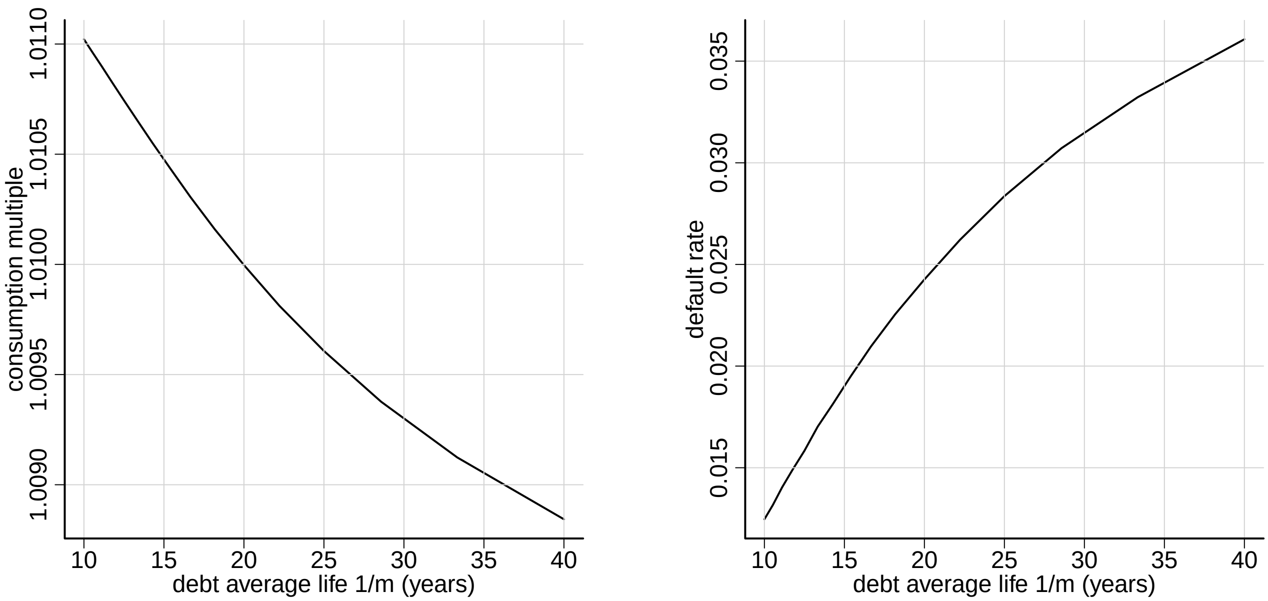
(a): Ergodic mean debt-to-income $\mathbb{E}(x)$ vs. γ | **(b):** Ergodic mean default rate λ_d vs. γ



Plot computed assuming $r = \kappa = 5\%$, $\delta = 7\%$, $\mu = 2\%$, $\sigma = 10\%$, $1/m = 20$ years, $\theta = 50\%$, $1 - \alpha = 4\%$.

Figure 17: Welfare gains and ergodic default rate

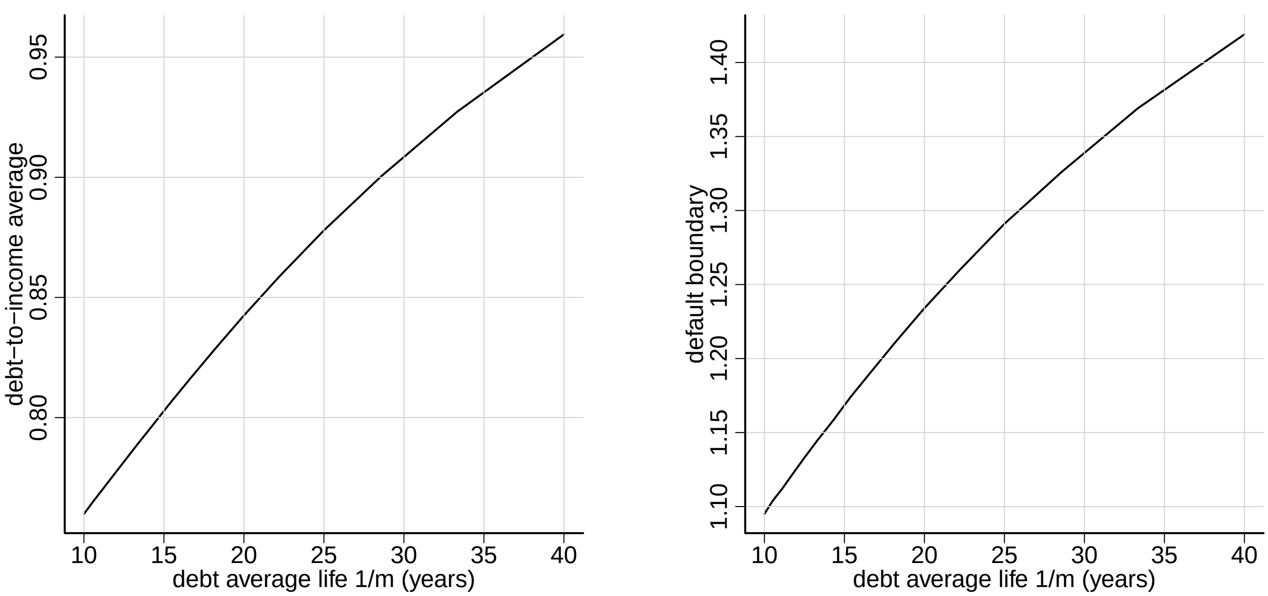
(a): Consumption multiple k_γ vs. $1/m$ **(b):** Default rate λ_d vs. $1/m$



Plot computed assuming $r = \kappa = 5\%$, $\delta = 7\%$, $\mu = 2\%$, $\sigma = 10\%$, $\gamma = 2$, $\theta = 50\%$, $1 - \alpha = 4\%$.

Figure 18: Debt-to-income ratio and default boundary

(a): Ergodic mean debt-to-income vs. $1/m$ **(b):** Default boundary $\bar{x}_\gamma$ vs. $1/m$



Plot computed assuming $r = \kappa = 5\%$, $\delta = 7\%$, $\mu = 2\%$, $\sigma = 10\%$, $\gamma = 2$, $\theta = 50\%$, $1 - \alpha = 4\%$.

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