Refinancing Frictions, Mortgage Pricing and Redistribution
Online Appendix

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1 Household's refinancing behavior

1.1 Value function V

Proof of Proposition 1. First, the household decision problem can be recast as follows:

$$V(x,c) := \inf_{k \in \mathcal{K}} \mathbb{E}_{x,c}\left[\int_0^{+\infty} e^{-\rho t}\left(c_t^{(k)}dt + \psi dN_t^{(k)}\right)\right], \tag{1.1}$$

$$\text{s.t.} \qquad dc_t^{(k)} = \left(m(x_t) - c_{t-}^{(k)}\right)\left(dN_t^{(k)} + dN_t^{(\nu)}\right),$$

where $\mathcal{K}$ is a set of progressively measurable intensity processes $k = \{k_t\}_{t \geq 0}$ such that $k_t \in [0, \chi]$ at all times. Using this definition, we first show that V must be increasing in c. Take $c' > c$ and an arbitrary intensity policy $k \in \mathcal{K}$. The difference in payoffs

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for such intensity policy k is:

$$\mathbb{E}_{x,c'}\left[\int_0^{+\infty} e^{-\rho t}\left(c_t^{(k)}dt + \psi dN_t^{(k)}\right)\right] - \mathbb{E}_{x,c}\left[\int_0^{+\infty} e^{-\rho t}\left(c_t^{(k)}dt + \psi dN_t^{(k)}\right)dt\right]$$
$$\geq \mathbb{E}_x\left[\int_0^{\tau} e^{-\rho t}(c'-c)dt\right] > 0,$$

where $\tau > 0$ a.s. is the first refinancing time under policy k. Taking the infimum over all admissible policies yields

$$\begin{aligned} V(x,c') &= \inf_{k\in\mathcal{K}} \mathbb{E}_{x,c'}\left[\int_0^{+\infty} e^{-\rho t}\left(c_t^{(k)}dt + \psi dN_t^{(k)}\right)dt\right] \\ &\geq \inf_{k\in\mathcal{K}} \mathbb{E}_{x,c}\left[\int_0^{+\infty} e^{-\rho t}\left(c_t^{(k)}dt + \psi dN_t^{(k)}\right)dt\right] = V(x,c) \end{aligned}$$

Thus V is increasing in c. Moreover, since $\chi < +\infty$, a reasoning by contradiction can show that V is in fact strictly increasing in c. Problem (1.1) is a standard stochastic control problem, for which standard results apply. For instance, for one-dimensional diffusions, and subject to some technical conditions on the operator $\mathcal{L}$, Strulovici and Szydlowski (2015)[1] provide for the value function V being twice continuously differentiable in x, and satisfying the following HJB equation:

$$(\rho+\nu)\,V(x,c) = c + \mathcal{L}V(x,c) + \nu V(x,m(x)) + \min_{k\in[0,\chi]}\left\{k\left(V(x,m(x)) + \psi - V(x,c)\right)\right\}$$

The optimal Markov control is $k^*(x,c) = \chi\mathbb{1}_{\{V(x,m(x))+\psi\leq V(x,c)\}}$. Since V is strictly increasing in c, this optimal policy can be re-written $k^*(x,c) = \chi\mathbb{1}_{\{c-m(x)\geq\theta(x)\}}$, for a rate gap cutoff $\theta(x)$ that satisfies

$$V(x, m(x)+\theta(x)) = V(x,m(x)) + \psi$$

$\theta(x)$ is well defined since V is continuous and strictly increasing in c. Reinjecting the optimal Markov control into the HJB equation satisfied by V yields the HJB equation in the main text. $\square$

1.2 Special case: m_t as a Brownian motion

Proof of Proposition 2. Assume that $m_t = \sigma B_t + m_0$, with B_t a Brownian motion. The household solves (1.1), where we note (with a slight abuse of notation) $V(m,c)$

[1] See also Fleming and Soner (2006); this latter article is not limited to one-dimensional diffusions, but includes additional – and more restrictive – conditions on the operator $\mathcal{L}$.

the value function. V can be simplified as follows

$$V(m,c) = \frac{c}{\rho} + \inf_{k\in\mathcal{K}} \mathbb{E}_{m,c}\left[\int_0^{+\infty} e^{-\rho t}\left[\left(c_t^{(k)} - c\right)dt + \psi dN_t^{(k)}\right]\right].$$

In other words, $V(m,c) = \frac{c}{\rho} + v(z)$, where $z = c - m$ is the rate gap and

$$\begin{aligned} v(z) &:= \inf_{k\in\mathcal{K}} \mathbb{E}_z\left[\int_0^{+\infty} e^{-\rho t}\left[\left(\psi - \frac{z_{t-}^{(k)}}{\rho}\right)dN_t^{(k)} - \frac{z_{t-}}{\rho}dN_t^{(\nu)}\right]\right] \\ dz_t^{(k)} &= -\sigma dB_t - z_{t-}^{(k)}\left(dN_t^{(k)} + dN_t^{(\nu)}\right). \end{aligned}$$

The value function v satisfies the following HJB:

$$(\rho+\nu+\chi)v(z) = \frac{\sigma^2}{2}v''(z) + \chi\min\left(v(z), v(0)+\psi-\frac{z}{\rho}\right) + \nu\left(v(0)-\frac{z}{\rho}\right)$$

Noting θ the rate gap above which the household finds it optimal to refinance when given the opportunity to do so, we must have

$$\begin{aligned} (\rho+\nu)v(z) &= \frac{\sigma^2}{2}v''(z) + \nu\left(v(0)-\frac{z}{\rho}\right) & z\le\theta \\ (\rho+\nu+\chi)v(z) &= \frac{\sigma^2}{2}v''(z) + \nu\left(v(0)-\frac{z}{\rho}\right) + \chi\left(v(0)+\psi-\frac{z}{\rho}\right) & z\ge\theta \end{aligned}$$

Introduce the constant $\eta_\chi := \frac{\sqrt{2(\rho+\nu+\chi)}}{\sigma}$. Note that $v(z) = O(z)$ as $z\to+\infty$ or as $z\to-\infty$. Thus we must have

$$v(z) = k_- e^{\eta_0(z-\theta)} + \frac{\nu\left(v(0)-\frac{z}{\rho}\right)}{\rho+\nu} \qquad z\le\theta \tag{1.2}$$

$$v(z) = k_+ e^{-\eta_\chi(z-\theta)} + \frac{\nu\left(v(0)-\frac{z}{\rho}\right)}{\rho+\nu+\chi} + \frac{\chi\left(v(0)+\psi-\frac{z}{\rho}\right)}{\rho+\nu+\chi} \qquad z\ge\theta \tag{1.3}$$

The constants k_-, k_+ must be such that v is continuously differentiable at $z=\theta$. Moreover, since we must have $\theta>0$, it must be the case that

$$v(0) = \frac{\rho+\nu}{\rho}k_- e^{-\eta_0\theta}$$

Taking θ as given, the requirement that v be continuously differentiable at $z = \theta$ yields a system of 2 equations in the 2 unknown k_-, k_+:

$$\begin{aligned} k_- + \frac{\nu}{\rho+\nu}\left[v(0) - \frac{\theta}{\rho}\right] &= k_+ + \frac{\nu}{\rho+\nu+\chi}\left[v(0) - \frac{\theta}{\rho}\right] + \frac{\chi}{\rho+\nu+\chi}\left[v(0) + \psi - \frac{\theta}{\rho}\right] \\ \eta_0 k_- - \frac{\nu}{\rho(\rho+\nu)} &= -\eta k_+ - \frac{\nu+\chi}{\rho(\rho+\nu+\chi)} \end{aligned}$$

Using our formula for $v(0)$, this system can be solved to yield:

$$\begin{aligned} k_- &= \frac{\chi}{(\rho+\nu+\chi)(\eta_0+\eta_\chi) - \chi\eta_\chi e^{-\eta_0\theta}}\left[\eta_\chi\left(\psi - \frac{\theta}{\rho+\nu}\right) - \frac{1}{\rho+\nu}\right] \\ k_+ &= \frac{-\chi}{(\rho+\nu+\chi)(\eta_0+\eta_\chi) - \chi\eta_\chi e^{-\eta_0\theta}}\left[\left(1 - \frac{\chi e^{-\eta_0\theta}}{\rho+\chi+\nu}\right)\left(\frac{1}{\rho+\nu}\right) + \eta_0\left(\psi - \frac{\theta}{\rho+\nu}\right)\right] \end{aligned}$$

At $z = \theta$, the household is indifferent between (a) continuing with the current mortgage, or (b) paying the fixed cost and refinancing. This means that

$$v(\theta) = v(0) + \psi - \frac{\theta}{\rho} \Rightarrow k_- = \frac{\rho}{\rho+\nu}v(0) + \psi - \frac{\theta}{\rho+\nu}$$

But since we know $v(0)$ as a function of k_-, this yields

$$k_- = k_- e^{-\eta_0\theta} + \psi - \frac{\theta}{\rho+\nu}$$

Using our formula for k_-, this yields, after some algebra, the implicit equation

$$e^{-\eta_0\theta} + \left[\eta_0 + \frac{\rho+\nu}{\chi}(\eta_0+\eta_\chi)\right]\theta = 1 + \left[\eta_0 + \frac{\rho+\nu}{\chi}(\eta_0+\eta_\chi)\right](\rho+\nu)\psi$$

Note $F(\theta)$ the left-hand-side of the above equation. Notice that F is convex, and $F'(0) > 0$, which means that F is strictly increasing for $\theta > 0$. Moreover, $F(0) = 1 < 1 + \left[\eta_0 + \frac{\rho+\nu}{\chi}(\eta_0+\eta_\chi)\right](\rho+\nu)\psi$, and $F(\theta) \to +\infty$ when $\theta \to +\infty$. In other words, the above equation admits a unique positive solution θ. Re-write the implicit equation satisfied by θ as follows:

$$\begin{aligned} e^{-\eta_0\theta} + (\eta_0 + \epsilon_\chi)\theta &= 1 + (\rho+\nu)\psi(\eta_0+\epsilon_\chi) \qquad (1.4) \\ \epsilon_\chi :&= \frac{(\rho+\nu)(\eta_0+\eta_\chi)}{\chi} \end{aligned}$$

Clearly, ϵ_χ is a positive and decreasing function of χ, converging to zero as $\chi \to +\infty$. Differentiate the above equation w.r.t. χ to obtain

$$\frac{\partial \theta}{\partial \chi} = \frac{((\rho+\nu)\psi - \theta)\frac{\partial \epsilon_\chi}{\partial \chi}}{\epsilon_\chi + \eta_0\left(1 - e^{-\eta_0 \theta}\right)} > 0,$$

with the last inequality following from $\frac{\partial \epsilon_\chi}{\partial \chi} < 0$ and $(\rho+\nu)\psi - \theta = \frac{e^{-\eta_0\theta}-1}{\eta_0+\epsilon_\chi} < 0$. Thus θ increases with χ. When $\chi \to +\infty$, the limiting value θ_∞ solves

$$e^{-\eta_0\theta_\infty} + \eta_0\theta_\infty = 1 + \eta_0\psi(\rho+\nu)$$

This delivers the ADL formula, using the Lambert function W:

$$\theta_\infty = \frac{1}{\eta_0}\left(1 + \eta_0\psi(\rho+\nu) + W\left(-\exp\left(-1 - \eta_0\psi(\rho+\nu)\right)\right)\right)$$

Instead, consider the case $\chi \to 0$. In that case, the optimal threshold converges to

$$\theta_0 = (\rho+\nu)\psi$$

We can also perform a Taylor expansion of (1.4) around $\theta = 0$, which allows us to obtain an approximation $\hat{\theta}$ of the value θ:

$$\frac{\eta_0^2}{2}\hat{\theta}^2 + \epsilon_\chi\hat{\theta} - (\rho+\nu)(\eta_0+\epsilon_\chi)\psi = 0$$

This allows us to conclude that the approximation $\hat{\theta}$ is equal to

$$\hat{\theta} = \sqrt{\frac{2}{\eta_0}\left(1 + \frac{\epsilon_\chi}{\eta_0}\right)(\rho+\nu)\psi + \left(\frac{\epsilon_\chi}{\eta_0^2}\right)^2} - \frac{\epsilon_\chi}{\eta_0^2}$$

One can also analyze the asymptotic behavior of the rate gap threshold as $\sigma \to +\infty$. Some algebra can quickly show that

$$\lim_{\sigma\to+\infty} \theta = (\rho+\nu)\psi\left[1 + \left(\frac{\chi}{\rho+\nu}\right)\frac{1}{1+\sqrt{\frac{\rho+\nu+\chi}{\rho+\nu}}}\right]$$

Thus, while the threshold diverges to $+\infty$ for perfectly attentive households (with $\chi = +\infty$), such threshold converges to a finite value when $\chi < +\infty$, as inattention attenuates the value of the refinancing option.

Finally, it is straightforward (but tedious) to verify that the optimal rate gap threshold θ is identical to that derived above if one were to assume a fixed cost upon moving. The intuition behind this result is straightforward: since there is equal probability that the household moves when the mortgage is in- or out-of-the-money (given the fact that the mortgage rate is a Brownian motion), the household's refinancing strategy does not change in the presence of fixed moving costs. □

1.3 Optimal threshold $\theta(x)$ vs. speed of mean-reversion

We assume that mortgage rates follow an Ornstein-Uhlenbeck process:

$$dm_t = -\kappa(m_t - \bar{m})dt + \sigma dB_t$$

This specification nests the special case of the Brownian motion studied in the main text, by setting $\kappa = 0$. We solve the stochastic control problem numerically with a finite difference scheme, and plot in Figure 1.1 the ergodic average threshold $\mathbb{E}\left[\theta(x_t)\right]$ and the ergodic average slope of the threshold $\mathbb{E}\left[\theta'(x_t)\right]$ as a function of the speed of mean-reversion κ. Our parameter choice includes (i) a subjective discount rate $\rho = 5\%$, (ii) a mortgage rate volatility $\sigma = 1\%$ (both (i) and (ii) being consistent with ADL), (iii) a moving plus amortization rate $\nu = 8.5\%$ (see main text), (iv) upfront closing costs $\psi = 2\%$ (consistent with ADL for their base case calibration under the assumption that the household has a mortgage balance of around \$200,000), and (v) an average rate $\bar{m} = 5\%$, consistent with the time-series average of mortgage rates from beginning 2000 until end 2021.

Figure 1.1 suggests that the average rate gap threshold $\mathbb{E}[\theta(x)]$ is mildly dependent on the half-life of the mortgage rate process—for our chosen parameters, it ranges from 45bps to 49bps. Most importantly though, moving from the pure random walk assumption to a mean-reverting process introduces some state dependence in the threshold $\theta(x)$.

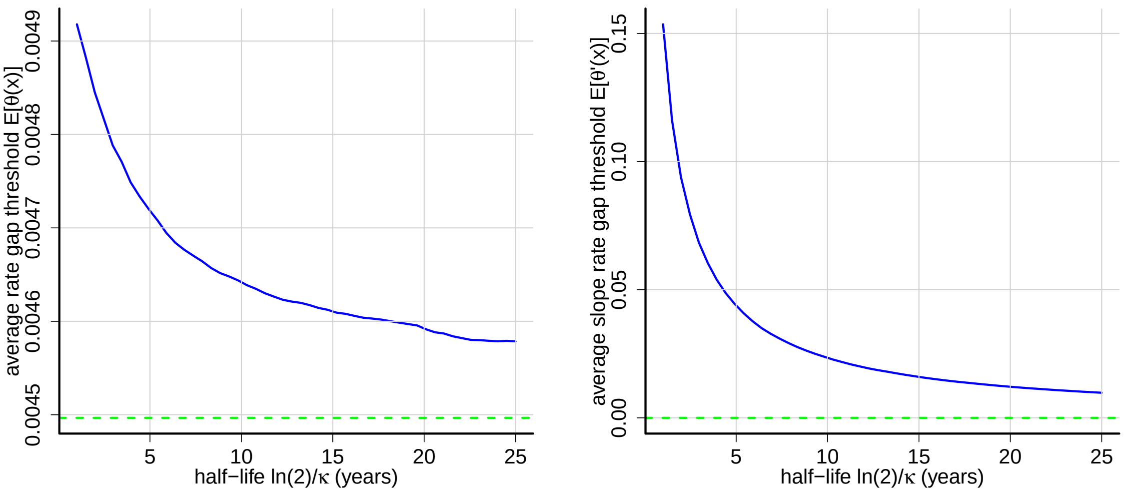

Figure 1.1: **Rate gap threshold θ vs. mortgage process half-life.** Rate gap threshold θ in the case of m_t following an OU process for various speeds of mean-reversion κ. Left plot shows the ergodic average threshold $\mathbb{E}\left[\theta(x_t)\right]$, while right plot shows the ergodic average slope of the threshold $\mathbb{E}\left[\theta'(x_t)\right]$. Horizontal dash green line represents the limit $\kappa \to 0$—i.e. when m_t is a Brownian motion. Figure computed for $\rho = 5\%$, $\chi = 31\%$, $\nu = 8.5\%$, $\psi = 2\%$, $\bar{m} = \mathbb{E}[m_t] = 5\%$ and $\sigma = 1\%$.

2 Mortgage market equilibrium

2.1 MPE existence and uniqueness in homogeneous case

Proof of Proposition 3. Discounted debt prices must be martingales, thus

$$r(x)P(x,c;\chi) = c - f + \mathcal{L}P(x,c;\chi) + \left(\nu + \chi \mathbb{1}_{\{c-m(x)\geq\theta(x)\}}\right)\left(1 - P(x,c;\chi)\right). \quad (2.1)$$

The function P, solution of (2.1), is implicitly dependent on a mortgage rate function $m(x)$, via the decision rule $\theta(x)$, which comes out of the household refinancing problem. It thus means that the equilibrium mortgage rate, implicitly defined via $P(x, m(x); \chi) = 1 + \pi$, is the outcome of a potentially complex fixed-point problem. Our proof has two steps; we first tackle the case $\pi = 0$, and then generalize to the case $\pi > 0$. In both cases, we assume no upfront closing costs (i.e. $\psi = 0$), and we assume that $r_t \in [\underline{r}, \bar{r}]$, with $0 \leq \underline{r} < \bar{r} < +\infty$, and $\chi < +\infty$. In that environment without upfront closing costs paid by households, the decision rule simplifies to $\theta(x) = 0$, in other words the optimal intensity solves $k^*(x,c) = \chi \mathbb{1}_{\{c\geq m(x)\}}$.

i. In this section, we restrict ourselves to the case where $\pi = 0$. To make further progress, we study the auxiliary problem

$$\tilde{P}(x,c;\chi) := \inf_{k\in\mathcal{K}} \mathbb{E}_x \left[\int_0^{+\infty} e^{-\int_0^t (r(x_s)+k_s+\nu)ds} \left(c - f + k_t + \nu\right) dt \right], \tag{2.2}$$

where $\mathcal{K}$ is defined in Section 1.1. The function $\tilde{P}$ does not depend, directly or indirectly, on any equilibrium object; in other words, one can view $\tilde{P}$ as the solution to a single-agent stochastic control problem. Arguments similar to those developed in Section 1.1 allow us to argue that $\tilde{P}$ is twice continuously differentiable in x, continuous and increasing in c, satisfying the HJB equation

$$\left(r(x)+\nu\right)\tilde{P}(x,c;\chi) = c - f + \nu + \mathcal{L}\tilde{P}(x,c;\chi) + \min_{k\in[0,\chi]} \left\{ k\left(1-\tilde{P}(x,c;\chi)\right)\right\}.$$

The optimal Markov control is $\tilde{k}(x,c) = \chi \mathbb{1}_{\left\{\tilde{P}(x,c;\chi)\geq 1\right\}}$. Since r_t is restricted to be on $\mathbb{R}_+$, we must have $\tilde{P}(x,0;\chi) < 1$. Similarly, since r_t is bounded above by $\bar{r}$, for c sufficiently high we must have $\tilde{P}(x,c;\chi) > 1$. Since $\tilde{P}$ is continuous and increasing in c, by the intermediate value theorem there must exist a unique real value $c = m(x)$ that satisfies

$$\tilde{P}(x, m(x); \chi) = 1 \tag{2.3}$$

Given this construction, and given that $\tilde{P}$ is monotone in c, the set of events $\{\tilde{P}(x_t,c;\chi) \geq 1\}$ is identical to the set of events $\{m(x_t) \leq c\}$. We can then verify that the auxiliary function $\tilde{P}$ is none other than the pricing function P, and the mortgage rate function $m(x)$ defined via (2.3) is unique and satisfies the equilibrium condition $P(x,m(x);\chi) = 1$.

ii. We now consider the case $\pi > 0$ — i.e. the case where mortgage origination triggers costs, borne by lenders and recouped via higher mortgage rates. In this section, we also assume that the latent state x is one-dimensional and $r(\cdot)$ is increasing. We will prove that there exists a unique monotone equilibrium in that case — i.e. a unique MPE in which the mortgage rate function is monotone increasing in x. Take an arbitrary x^*, and define $\tau_{x^*,\chi}$ as a stopping time with arrival intensity $\nu + \chi\mathbb{1}_{\{x_t\leq x^*\}}$.[2] As will be seen shortly, x^* represents the latent state that was prevalent the last time a household refinanced. Consider the interest-only "IO" and principal-only "PO" net present values, defined via

$$IO(x;\chi) := \mathbb{E}_x\left[\int_0^{\tau_{x,\chi}} e^{-\int_0^t r_s ds} dt\right] \qquad PO(x;\chi) := \mathbb{E}_x\left[e^{-\int_0^{\tau_{x,\chi}} r_s ds}\right]$$

[2] Formally, if ω is a (unit mean) exponentially distributed random variable and if we introduce the compensator $\Lambda_t = \int_0^t (\nu + \chi\mathbb{1}_{\{x_s\leq x^*\}})ds$, then the stopping time $\tau_{x^*,\chi}$ is the (random) time that satisfies $\Lambda_{\tau_{x^*,\chi}} = \omega$.

These objects represents, respectively, the valuation of an IO and a PO whenever the latent state variable is x, and whenever the prepayment time is driven by a point process with (time-varying) intensity $\nu + \chi \mathbb{1}_{\{x_t \leq x\}}$. Introduce the function m, defined via

$$m(x) := f + \frac{1 - PO(x;\chi)}{IO(x;\chi)} + \frac{\pi}{IO(x;\chi)}. \tag{2.4}$$

m is continuous in x. We argue that m is a monotone increasing function of x, and that a monotone equilibrium exists, in which $m(x)$ is the equilibrium mortgage market interest rate. Consider first the special case $\pi = 0$. In that case, we know from the previous section (i) that an equilibrium exists and is unique. Since the objective in problem (2.2) is decreasing in x, it must be the case that the function $\tilde{P}$ defined in (2.2) is decreasing in x, which must mean that the equilibrium mortgage rate, when $\pi = 0$, is monotone increasing in x. In that case, the mortgage rate function must correspond to that defined in (2.4) (with $\pi = 0$)—this is the case since

$$\begin{aligned}\tilde{P}\left(x, m(x); \chi\right) = 1 &= \mathbb{E}_x \left[\int_0^{\tau_{x,\chi}} e^{-\int_0^t r_s ds} (m(x) - f) dt + e^{-\int_0^{\tau_{x,\chi}} r_s ds} \right] \\ &= (m(x) - f) IO(x;\chi) + PO(x;\chi),\end{aligned}$$

which directly implies (2.4) for $\pi = 0$. As $m(x)$ is increasing when $\pi = 0$, it must be the case that $\left(1 - PO(x;\chi)\right)/IO(x;\chi)$ is increasing in x. For $\pi > 0$, we additionally need to show that $1/IO(x;\chi)$ is increasing in x. To this end, note that for $x_1 < x_2$, we must always have

$$\mathbb{E}_{x_2} \left[\int_0^{\tau_{x_2,\chi}} e^{-\int_0^t r_s ds} dt \right] \leq \mathbb{E}_{x_1} \left[\int_0^{\tau_{x_2,\chi}} e^{-\int_0^t r_s ds} dt \right] \leq \mathbb{E}_{x_1} \left[\int_0^{\tau_{x_1,\chi}} e^{-\int_0^t r_s ds} dt \right].$$

The first inequality above stems from the fact that if the initial interest rate is $r(x_1)$, the full time path of future interest rates is below that which would be relevant if the initial interest rate was $r(x_2)$. The second inequality stems from the fact that, for a given starting level of the latent state x_1, we must have the stopping time inequality $\tau_{x_2,\chi} \leq \tau_{x_1,\chi}$ almost surely. In other words, $IO(x;\chi)$ must be decreasing in x. This allows us to conclude that m, defined in (2.4), is monotone increasing in x. Given this observation, we must have an equilibrium in which m is the mortgage rate, since m must satisfy

$$1 + \pi = \mathbb{E}_x \left[\int_0^{\tau_{x,\chi}} e^{-\int_0^t r_s ds} (m(x) - f) dt + e^{-\int_0^{\tau_{x,\chi}} r_s ds} \right]$$

That equilibrium is unique, since we showed its existence by construction. In other words, in any monotone equilibrium, it must be the case that the mortgage rate function satisfies (2.4).

$\square$

2.2 Comparative statics

Proof of Proposition 4. Consider first the case $\pi = 0$. Since $P = \tilde{P}$ can be defined via equation (2.2), it must be the case that P is decreasing in χ. Thus, the mortgage rate $m(x)$, defined implicitly via (2.3), is increasing in χ, whenever $\pi = 0$. Consider then the case where $\pi > 0$, and where the latent state x is one-dimensional and $r(\cdot)$ is increasing. Given our conclusion for the case $\pi = 0$, it must be the case that $(1 - PO(x;\chi))/IO(x;\chi)$ is increasing in χ. Define (with a slight abuse of notation) $IO(x, x^*; \chi) := \mathbb{E}_x \left[\int_0^{\tau_{x^*,\chi}} e^{-\int_0^t r_s ds} dt\right]$, which solves the PDE

$$\left(r(x) + \nu + \chi \mathbb{1}_{\{x \leq x^*\}}\right) IO(x, x^*; \chi) = 1 + \mathcal{L} IO(x, x^*; \chi)$$

Differentiate this equation w.r.t. χ to obtain

$$\left(r(x) + \nu + \chi \mathbb{1}_{\{x \leq x^*\}}\right) \partial_\chi IO(x, x^*; \chi) = -\mathbb{1}_{\{x \leq x^*\}} IO(x, x^*; \chi) + \mathcal{L} \partial_\chi IO(x, x^*; \chi)$$

Thus, $\partial_\chi IO(x, x^*; \chi)$ admits the integral representation

$$\partial_\chi IO(x, x^*; \chi) = -\mathbb{E}_x \left[\int_0^{\tau_{x^*,\chi}} e^{-\int_0^t r_s ds} \mathbb{1}_{\{x_t \leq x^*\}} IO(x_t, x^*; \chi) dt\right] < 0$$

Thus, $IO(x;\chi)$ is monotone decreasing in χ. This must mean that the mortgage rate function, defined via (2.4), is increasing in χ, whenever $\pi > 0$. $\square$

2.3 Small fixed costs

Proof of Proposition 5. Suppose an environment where upfront closing costs ψ are small, and where the gain on sale is $\pi = 0$. Postulate an MPE in which mortgage prices P, households' optimal rate gap threshold θ, and equilibrium mortgage rates m can be written:

$$\begin{aligned}
P(x, c) &\underset{\psi \to 0}{=} P_0(x, c) + \psi P_1(x, c) + o(\psi) \\
\theta(x) &\underset{\psi \to 0}{=} \theta_0(x) + \psi \theta_1(x) + o(\psi) \\
m(x) &\underset{\psi \to 0}{=} m_0(x) + \psi m_1(x) + o(\psi),
\end{aligned}$$

where P_0, θ_0, m_0 are respectively mortgage prices, households' optimal rate gap threshold, and equilibrium mortgage rates in the MPE where $\psi = 0$. We know from our previous analysis that $\theta_0 = 0$. Consider the PDE satisfied by mortgage prices:

$$\left(r + \nu + \chi 1_{\{c-m(x)\geq\theta(x)\}}\right) P(x,c) = c - f + \nu + \chi 1_{\{c-m(x)\geq\theta(x)\}} + \mathcal{L}P(x,c)$$

Let $\epsilon_1(x) := m_1(x) + \theta_1(x)$, we then have

$$\begin{aligned} 1_{\{c-m(x)\geq\theta(x)\}} = 1_{\{c-m_0(x)\geq\psi\epsilon_1(x)\}} &= 1_{\{c-m_0(x)\geq 0\}} + 1_{\{c-m_0(x)\geq\psi\epsilon_1(x)\}} - 1_{\{c-m_0(x)\geq 0\}} \\ &= 1_{\{c-m_0(x)\geq 0\}} - \frac{\epsilon_1(x)}{|\epsilon_1(x)|} 1_{\{|c-m_0(x)|\in[0,\psi|\epsilon_1(x)|]\}} \end{aligned}$$

This allows us to write the zero-order expansion of the mortgage price as follows:

$$\left(r + \nu + \chi 1_{\{c-m_0(x)\geq 0\}}\right) P_0(x,c) = c - f + \nu + \chi 1_{\{c-m_0(x)\geq 0\}} + \mathcal{L}P_0(x,c)$$

For the first order expansion of the mortgage price, first note that we have, whenever c is in a neighbourhood of $m_0(x)$:

$$P_0(x,c) \underset{c\to m_0(x)}{=} 1 + (c - m_0(x))\, \partial_c P_0\left(x, m_0(x)\right) + o\left(|c - m_0(x)|\right),$$

where we have used the equilibrium condition $P_0\left(x, m_0(x)\right) = 1$. Note also that whenever $|c - m_0(x)| \in [0, \psi|\epsilon_1(x)|]$, there exists a $k_\psi(x,c) \in [0,1]$ s.t. $c = m_0(x) + \psi k_\psi(x,c)\epsilon_1(x)$. Whenever $|c - m_0(x)| \notin [0, \psi|\epsilon_1(x)|]$, set $k_\psi(x,c) = 0$. The first order correction term then satisfies:

$$\begin{aligned} \left(r + \nu + \chi 1_{\{c-m_0(x)\geq 0\}}\right) P_1(x,c) &= \frac{\chi\epsilon_1(x)}{|\epsilon_1(x)|} 1_{\{c-m_0(x)\in[0,\psi|\epsilon_1(x)|]\}} \frac{P_0(x,c) - 1}{\psi} + \mathcal{L}P_1(x,c) \\ &= \chi k_\psi(x,c)|\theta_1(x) + m_1(x)|\partial_c P_0\left(x, m_0(x)\right) + \mathcal{L}P_1(x,c) \end{aligned}$$

Finally, as $\psi \to 0$, $k_\psi(x,c)$ is a bounded function that is non-zero on an interval with measure proportional to ψ. In other words, the first order correction term P_1 satisfies

$$\left(r + \nu + \chi 1_{\{c-m_0(x)\geq 0\}}\right) P_1(x,c) = \mathcal{L}P_1(x,c)$$

Since the source term is identically zero, we conclude that $P_1(x,c) = 0$. Lastly, the break-even condition can be written:

$$P\left(x, m(x)\right) = 1 = P_0\left(x, m_0(x)\right) + \psi P_1\left(x, m_0(x)\right) + \psi m_1(x)\partial_c P_0\left(x, m_0(x)\right) + o(\psi)$$

Thus, we have

$$m_1(x) = -\frac{P_1\left(x, m_0(x)\right)}{\partial_c P_0\left(x, m_0(x)\right)}$$

Since $P_1(x,c) = 0$, we can conclude that the first order correction term $m_1(x) = 0$. In other words, whenever $\pi = 0$, fixed costs have no impact (at the first order) on the equilibrium mortgage rate function. This analysis is supported by our numerical computations.

□

2.4 Infinite dimensional problem with heterogeneity

In this section, we discuss the key mathematical equations characterizing the pooling MPE. As a reminder, $H(\chi)$ denotes the cumulative distribution over types (with associated density h), while F_t denotes the joint cumulative distribution over outstanding coupon rates c and types χ in the population at time t (with associated joint density $f_t(c,\chi)$). Since types are a permanent household attribute, we must have

$$\int_c f_t(c,\chi)dc = h(\chi). \tag{2.5}$$

Consider then the density f_t. It evolves endogenously over time with idiosyncratic mortgage refinancing decisions, which, aggregated using a weak law of large numbers, lead to locally deterministic movements in f_t. The Kolmogorov Forward Equation ("KFE") that describes these changes is then, for $c \neq m(S)$:

$$df_t(c,\chi) = -\left(\nu + \chi \mathbb{1}_{\{c \geq m(S_t)\}}\right) f_t(c,\chi)dt, \qquad c \neq m(S). \tag{2.6}$$

The density f_t, between t and $t+dt$, looses mass at rate ν for $c < m(S_t)$, and at the higher rate rate $\nu + \chi$ for $c \geq m(S_t)$, as households strategically refinance. This equation holds everywhere except at $c = m(S_t)$, a state at which refinancing and moving households are being "reinjected"; the relevant equation in that case is

$$\lim_{c \uparrow m(S_t)} \partial_c f_t(c,\chi) - \lim_{c \downarrow m(S_t)} \partial_c f_t(c,\chi) = \nu h(\chi) + \chi \int_{m(S_t)}^{+\infty} f_t(c,\chi)dc. \tag{2.7}$$

The right-hand-side of this equation is the flux of households exogenously moving at rate ν and the flux of type-χ households refinancing in the time interval $[t, t+dt]$, while the left-hand-side is the kink in the density at $c = m(S)$ induced by the reinjection of such households at that particular point of the state space.

Let $P(S,c;\chi)$ be the *shadow* price of a mortgage with coupon c, conditional on knowing that the related household has attention rate χ. The shadow price solves the following infinite dimensional Feynman-Kac equation, which takes into account

(i) changes in the distribution f_t, and (ii) the behavior of type-χ households:

$$r(x)P(S,c;\chi) = c + \mathcal{L}P(S,c;\chi) + \left(\nu + \chi\mathbb{1}_{\{c\geq m(S)\}}\right)[1 - P(S,c;\chi)] \\ + \int \mathcal{T}[f](c,\chi)\frac{\delta P}{\delta f(c,\chi)}dcd\chi, \quad (2.8)$$

with $\delta P/\delta f$ the functional derivative of P w.r.t. f at (c,χ) and the operator $\mathcal{T}$ defined via

$$\mathcal{T}[f](c,\chi) = -\left(\nu + \chi\mathbb{1}_{\{c>m(S)\}}\right)f(c,\chi) \quad (2.9)$$

See Achdou, Buera, Lasry, Lions, and Moll (2014) for another example of such infinite-dimensional PDE in the context of consumption-savings models in incomplete markets with aggregate shocks.

2.5 Approximate pooling MPE existence and uniqueness

Proof of Proposition 6. We establish the existence and uniqueness of the approximate pooling MPE using a method similar to Section 2.1 for the case $\pi > 0$. To that effect, consider the dynamic system $(x_t, x^*_{t,\chi})$, where

$$dx_t = \mu(x_t)dt + \sigma(x_t)dB_t$$
$$dx^*_{t,\chi} = \left(x_t - x^*_{t-,\chi}\right)dN^\chi_t,$$

where N^χ_t is a point process with arrival intensity $\nu + \chi\mathbb{1}_{\{x_t\leq x^*_{t-,\chi}\}}$. This dynamic system admits a generator $\mathcal{H}_\chi$ defined for any smooth function $\phi(x,x^*)$ via

$$\mathcal{H}_\chi\phi(x,x^*) = \mathcal{L}\phi(x,x^*) + \left(\nu + \chi\mathbb{1}_{\{x\leq x^*\}}\right)(\phi(x,x) - \phi(x,x^*))$$

The eigen-function (associated with the eigen-value zero) of the adjoint of the operator $\mathcal{H}_\chi$ gives us the stationary density $f_\infty(x,x^*|\chi)$ of the dynamic system $(x_t, x^*_{t,\chi})$. Introduce then the distribution g, either the unconditional one defined via

$$g(\chi) = \frac{h(\chi)\int_x\left[\left(\nu + \chi\int_{x^*\geq x} f_\infty(x^*|x,\chi)dc\right)\right]f_\infty(x)dx}{\int_\chi h(\chi)\int_x\left(\nu + \chi\int_{x^*\geq x} f_\infty(x^*|x,\chi)dc\right)f_\infty(x)d\chi dx}, \quad (2.10)$$

or the conditional one defined via

$$g(\chi|x) = \frac{h(\chi)\left(\nu + \chi\int_{x^*\geq x} f_\infty(x^*,x|\chi)dc\right)}{\int_\chi h(\chi)\left(\nu + \chi\int_{x^*\geq x} f_\infty(x^*,x|\chi)dc\right)d\chi}. \quad (2.11)$$

Define the candidate mortgage rate $m(x;G)$ via

$$m(x;G) := f + \frac{1+\pi - \mathbb{E}^G\left[PO(x;\chi)\right]}{\mathbb{E}^G\left[IO(x;\chi)\right]}, \tag{2.12}$$

If the function $m(x;G)$ is increasing in x, a monotone approximate pooling MPE must exist, and this equilibrium is unique amongst all monotone equilibria. Note that m must satisfy

$$1+\pi = \mathbb{E}^G\left[\mathbb{E}_x\left[\int_0^{\tau_{x,\chi}} e^{-\int_0^t r_s ds}(m(x)-f)dt + e^{-\int_0^{\tau_{x,\chi}} r_s ds}\right]\right]$$

Consider then the price $\bar{P}_G\left(x, m(x^*)\right)$ of a mortgage with coupon $m(x^*)$,

$$\bar{P}_G(x, m(x^*)) := \mathbb{E}^G\left[\mathbb{E}_x\left[\int_0^{\tau_{x,\chi}} e^{-\int_0^t r_s ds}(m(x^*)-f)dt + e^{-\int_0^{\tau_{x,\chi}} r_s ds}\right]\right],$$

then clearly if m is increasing, $\bar{P}_G$ must be increasing in x^*, with $\bar{P}_G\left(x^*, m(x^*)\right) = 1+\pi$ – in other words the equilibrium conditions are satisfied. □

2.6 Integral representation of $\bar{P}_G$ for unconditional $G(\chi)$

Proof of Proposition 7. (2.1) holds for all χ, and thus, taking expectations w.r.t. the unconditional issuance type distribution $G\left(\chi\right)$, we have

$$\begin{aligned} r(x)\bar{P}_G\left(x,c\right) = c - f - \mathbb{1}_{\{m(x)\le c\}}\mathbb{C}\mathrm{ov}^G\left(\chi, P\left(x,c;\chi\right)\right) + \mathcal{L}\bar{P}_G\left(x,c\right) \\ + \left(\nu + \bar{\chi}_G\mathbb{1}_{\{m(x)\le c\}}\right)\left(1-\bar{P}_G\left(x,c\right)\right) \end{aligned} \tag{2.13}$$

One can then use Feynman-Kac to conclude that $\bar{P}_G$ admits the integral representation in Proposition 7. □

2.7 Invariance of lowest attainable mortgage rate

Proof of Proposition 8. Under the assumption that x is uni-dimensional and that $r(\cdot)$ is monotone increasing, call $\underline{x}$ the lowest bound for x. The monotone approximate pooling MPE implies m is increasing in x, and thus $m(\underline{x})$ must be the lowest attainable mortgage rate. Then we have

$$P\left(x, m(\underline{x}); \chi\right) = P\left(x, m(\underline{x}); \chi'\right), \qquad \forall \chi, \chi',$$

as χ only influences the refinancing channel, and whenever households have locked in the lowest possible rate $c = m(\underline{x})$, we have $c \le m(x_t)$ at all future date t regardless of type χ. Thus, from the break-even condition $\bar{P}(x, m(x)) = 1 + \pi$, we have

$$P(\underline{x}, m(\underline{x}); \chi) = 1 + \pi, \qquad \forall \chi.$$

$m(\underline{x})$ is thus invariant to the distribution G, and thus the distribution H. □

3 Policy evaluations and counterfactuals

3.1 Construction of the auto-RM

We first argue that the auto-RM market rate is a reference rate computed by looking at debt instruments traded in the market and prepayable at any time, with a call premium π. Indeed, note $P^*(x, c)$ the price of such a prepayable instrument with coupon c when the latent aggregate state is x:

$$P^*(x, c) := \inf_{\tau} \mathbb{E}_x \left[\int_0^{\tau \wedge \tau_\nu} e^{-\int_0^t r(x_s) ds} (c - f)\, ds + 1_{\{\tau \le \tau_\nu\}} (1 + \pi) e^{-\int_0^\tau r(x_s) ds} + 1_{\{\tau \ge \tau_\nu\}} e^{-\int_0^{\tau_\nu} r(x_s) ds} \right],$$

where τ_ν is a Poisson time with arrival rate ν. This optimal stopping problem is a free-boundary problem, with an endogeneous boundary $m^*(x)$ to be determined. The variational inequality, valid for any x, is

$$\max \left\{ -(\nu + r(x)) P^*(x, c) + c - f + \nu + \mathcal{L} P^*(x, c), P^*(x, c) - (1 + \pi) \right\} = 0$$

The HJB (i.e. the left hand side of the above inequality) holds in the continuation region $c \le m^*(x)$, while the equality $P^*(x, c) = 1 + \pi$ holds at the boundary of the continuation region $c = m^*(x)$, where $m^*(x)$ is the "auto-RM rate". The optimality condition for the stopping time τ takes the form of the smooth pasting condition $\partial_x P^*(x, m^*(x)) = 0$. The price function $P^*(x, c)$ satisfies $P^*(x, c) \le 1 + \pi$ for any coupon c and latent state x. P^* is increasing in c, and of course $P^*(x, m^*(x)) = 1 + \pi$. Note then that P^* is the limit, as $\chi \to +\infty$, of the following problem

$$\begin{aligned} \hat{P}(x, c; \chi) :&= \inf_{k \in \mathcal{K}_\chi} \mathbb{E}_x \left[\int_0^{\tau_k \wedge \tau_\nu} e^{-\int_0^t r(x_s) ds} (c - f)\, ds \right. \\ &\qquad \left. + 1_{\{\tau_k \le \tau_\nu\}} (1 + \pi) e^{-\int_0^{\tau_k} r(x_s) ds} + 1_{\{\tau_k \ge \tau_\nu\}} e^{-\int_0^{\tau_\nu} r(x_s) ds} \right] \\ &= \inf_{k \in \mathcal{K}_\chi} \mathbb{E}_x \left[\int_0^{+\infty} e^{-\int_0^t (r(x_s) + \nu + k_s) ds} (c - f + \nu + k_t (1 + \pi))\, ds \right], \end{aligned}$$

where $\mathcal{K}_\chi$ is the set of progressively measurable processes $\{k_t\}_{t \ge 0}$ so that $k_t \in [0, \chi]$ for all t, and τ_k, in the first equation, is a Poisson time with jump intensity k_t. The auto-RM rate is thus a reference rate that can be computed by looking at debt instruments traded in the market, and that are prepayable at any time at $1 + \pi$. These prepayable debt instruments, when issued, have a price and market value of $1 + \pi$, and a fair coupon equal to $m^*(x)$, the reference rate for the auto-RM.[3] Households are then locked into that auto-RM instrument, pay the floating rate $m^*(x_t)$ at all times, up to the point where they move. At such time, they prepay the mortgage balance \$1, and are forced to refinance into a new mortgage. Upon taking a new mortgage, households receive proceeds \$1 from lenders, but given that the loan pays a reference rate $m^*(x)$, the market value of such loan is equal to $1+\pi$, meaning that lenders can recoup their origination costs. Households then pay the floating rate $m^*(x_t)$ until the time they move and sell their house. By construction, the reference rate $m^*(x_t)$ satisfies

$$m^*(x_t) = \inf_{t \geq s \geq 0} m^*(x_s)$$

3.2 Auto-RM vs. short rates

Proof of Proposition 9. We consider the case $\pi \geq 0$ — i.e. the case where mortgage origination costs are potentially incurred, and recouped by lenders via higher mortgage rates. As discussed in Appendix 3.1, the price P^* of the auto-RM solves

$$P^*(x, c) = \inf_{\tau} \mathbb{E}_x \left[\int_0^{\tau} e^{-\int_0^t (r(x_s)+\nu) ds} \left(c - f + \nu\right) ds + (1+\pi) e^{-\int_0^{\tau} (r(x_s)+\nu) ds} \right].$$

Now assume for a second that there exists a latent state $\hat{x}$ so that $r(\hat{x}) > m(\hat{x}) - f$. Assume at time $t = 0$, $x_0 = \hat{x}$, and consider a stopping strategy $T = \inf\{t \geq 0 : r(x_t) = m(\hat{x}) - f\}$. Clearly, since $r(x_0) = r(\hat{x}) > m(\hat{x}) - f$ and since x has continuous sample path, $T > 0$ a.s. We then have the following set of inequalities

$$\begin{aligned} 1 + \pi = P^*(\hat{x}, m(\hat{x})) &= \inf_{\tau} \quad \mathbb{E}_{\hat{x}} \left[\int_0^{\tau} e^{-\int_0^t (r(x_s)+\nu) ds} (m(\hat{x}) - f + \nu) dt + (1+\pi) e^{-\int_0^{\tau} (r(x_s)+\nu) ds} \right] \\ &\leq \mathbb{E}_{\hat{x}} \left[\int_0^{T} e^{-\int_0^t (r(x_s)+\nu) ds} (m(\hat{x}) - f + \nu) dt + (1+\pi) e^{-\int_0^{T} (r(x_s)+\nu) ds} \right] \\ &< 1 + \pi, \end{aligned}$$

where the last inequality follows since for $t < T$, we must have $r(x_t) > m(\hat{x}) - f$. This is the contradiction we were looking for. $\square$

3.3 Auto-RM impact on initial debt-to-income ratio

[3] Given the nature of Brownian motions, these prepayable instruments have, at the time of issuance, zero duration.

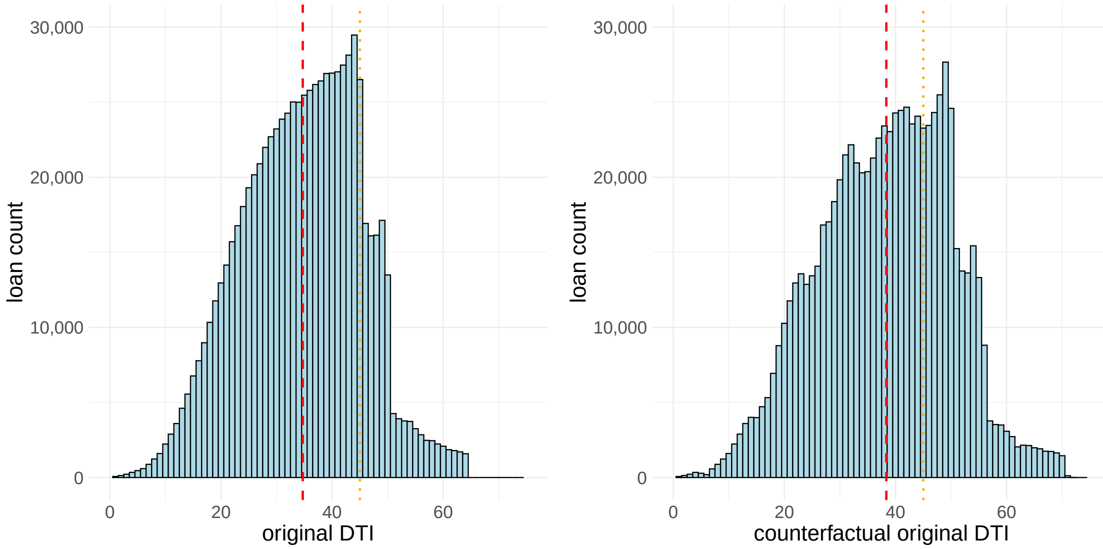

Figure 3.1: **DTI distribution and counterfactual DTI distribution.** Left figure shows the DTI distribution in the SFLP data. Right figure shows the counterfactual DTI distribution if mortgage rates were higher than those actually realized, with a difference corresponding to the ergodic average difference between (a) mortgage rates in the approximate pooling MPE and (b) mortgage rates in the auto-RM equilibrium. Vertical red dashed lines indicate average DTI ratio, while orange dotted lines indicate the 43% DTI limit.

4 Household attention in mortgage refinancing data

4.1 Data

We rely on information from Equifax Credit Risk Insight Servicing McDash ("CRISM"). This monthly-frequency data-set covers the period from May 2005 until December 2017. It contains unique borrower IDs, mortgage IDs, a prepayment indicator if a loan prepaid in a given month, the original coupon rate on the loan, its current principal balance, as well as the current FICO score of the related borrower. We build an indicator describing the type of prepayment (rate refinancing, cash-out refinancing or moves), and a measure of the current combined loan-to-value ratio (thereafter, "CTLV") using house price data from Corelogic. We construct, for each household and each month, the effective mortgage market rate available to a household, by regressing observed contractual rates on characteristics. This allows us to construct the rate gap – i.e. the difference between (i) the mortgage coupon and (ii) the household effective mortgage market rate. Our data-set allows us to track a borrower and their different mortgages through time. It contains 20,094,230 loan-month-borrower observations, with 246,330 unique borrower IDs.

For some of our econometric work, we also leverage the single-family loan performance ("SFLP") data-set from Fannie-Mae. While CRISM allows us to track individual households across loans, SFLP only allows us to track monthly mortgage performance data for a sample of conforming loans originated between January 2000 and December 2021. This means that the SFLP data cannot distinguish refinancing from other types of prepayment. However, this data is nevertheless useful since it contains covariates which are absent in CRISM — for instance the identity of the original lender and of the mortgage servicer.

4.2 Estimating the attention distribution H

For our clustering algorithm, we fix N, the number of groups. Whether a household belongs to one group or another is determined via maximum likelihood — we note $\alpha : \{1, \dots, N_h\} \to \{1, \dots, N\}$ the group assignment function. It is easier to state this optimization in terms of attention probability per month p, rather than in terms of attention rate χ.

For each household i, we associate a binomial random variable where the number of "successes" is the number of observed prepayment events, and the number of "trials" is the number of observed monthly periods. Let s_i^+ and t_i^+ (resp. s_i^- and t_i^-) be the number of successes and trials when rate gaps satisfy $gap_{it} > \theta$ (resp. $gap_{it} \leq \theta$). Let y_{it} be an indicator of household i prepaying at time t, and define the gap-dependent monthly prepayment probabilities

$$p^+_{\alpha(i)} := 1 - \exp\big(-(\nu + \chi_{\alpha(i)})dt\big) \qquad\qquad p^-_{\alpha(i)} := 1 - \exp\left(-\nu dt\right)$$

Then, the log-likelihood of observing y_{it} is given by

$$\begin{aligned}\mathcal{L}(y_{it}) = y_{it} \Big(&\mathbb{1}_{\{gap_{it} > \theta\}} \log p^+_{\alpha(i)} + \mathbb{1}_{\{gap_{it} \leq \theta\}} \log p^-_{\alpha(i)}\Big) \\ &+ (1 - y_{it}) \Big(\mathbb{1}_{\{gap_{it} > \theta\}} \log(1 - p^+_{\alpha(i)}) + \mathbb{1}_{\{gap_{it} \leq \theta\}} (1 - \log p^-_{\alpha(i)})\Big) \end{aligned} \tag{4.1}$$

Let $t_i := t_i^- + t_i^+$ be the total number of months household i is in the sample. Then the log-likehood of observing a sequence $(y_{i1}, ..., y_{it_i})$ is given by

$$\begin{aligned}\mathcal{L}(y_{i1}, ..., y_{it_i}) &= \sum_{t=1}^{t_i} \big(y_{it} \mathbb{1}_{\{gap_{it} > \theta\}}\big) \log p^+_{\alpha(i)} + \sum_{t=1}^{t_i} \big(y_{it} \mathbb{1}_{\{gap_{it} \leq \theta\}}\big) \log p^-_{\alpha(i)} \\ &+ \sum_{t=1}^{t_i} \big((1 - y_{it}) \mathbb{1}_{\{gap_{it} > \theta\}}\big) \log(1 - p^+_{\alpha(i)}) + \sum_{t=1}^{t_i} \big((1 - y_{it}) \mathbb{1}_{\{gap_{it} \leq \theta\}}\big) (1 - \log p^-_{\alpha(i)}) \\ &= s_i^+ \log p^+_{\alpha(i)} + s_i^- \log p^-_{\alpha(i)} + (t_i^+ - s_i^+) \log(1 - p^+_{\alpha(i)}) + (t_i^- - s_i^-) \log(1 - p^-_{\alpha(i)}) \end{aligned} \tag{4.2}$$

where we used the identities

$$s_i^+ = \sum_{t=1}^{t_i} (1 - y_{it}) \mathbb{1}_{\{gap_{it} > \theta\}}, \qquad s_i^- = \sum_{t=1}^{t_i} y_{it} \mathbb{1}_{\{gap_{it} \leq \theta\}} \tag{4.3}$$

$$t_i^+ = \sum_{t=1}^{t_i} \mathbb{1}_{\{gap_{it} > \theta\}}, \qquad t_i^- = \sum_{t=1}^{t_i} \mathbb{1}_{\{gap_{it} \leq \theta\}} \tag{4.4}$$

Of course, there are several sequences $(y_{i1}, ..., y_{it_i})$ that results in the same $(s_i^+, s_i^-, t_i^+, t_i^-)$. Thus, for the total log-likehoood we have to sum over all the appropriate permutation, which results in a factor similar to $\binom{a}{b}$, the binomial factor. However, we note that this factor is independent of the choices of $p^+_{\alpha(i)}$ and $p^-_{\alpha(i)}$, the per-month probabilities, and thus does not affect our ML estimation. Given values $\{s_i^+, s_i^-, t_i^+, t_i^-\}_{i \leq N_h}$, we want to estimate the maximum likelihood that those observations were generated by $N + 1$ prepayment probabilities $(p_0, ..., p_N)$, where p_0 represents the prepayment probability conditional on $gap \leq \theta$, while $p_1, ..., p_N$ represent N prepayment probabilities conditional on $gap > \theta$. Of course we need to insure that $p_k \geq p_0$, for $k \geq 1$. If $\mathbf{P} := \{(p_0, ..., p_N) : p_k \in [0, 1]\, \forall k, p_k \geq p_0 \forall k \geq 1\}$, then the maximum log-likelihoood of the data, for a given $(p_0, ..., p_N) \in \mathbf{P}$, is

$$L(\mathbf{P}) = \sum_{i=1}^{N_h} \max_{p^+ \in \{p_1, ..., p_N\}} \big[s_i^+ \log p^+ + s_i^- \log p_0 \\ + (t_i^+ - s_i^+) \log(1 - p^+) + (t_i^- - s_i^-) \log(1 - p_0) \big\} \tag{4.5}$$

To get the MLE, we maximize $L(\mathbf{P})$ over the set $\mathbf{P}$.

4.3 Attention rate distribution

Table 4.1 reports our point estimate and standard errors for the discrete distribution H of attention types in our sample of borrowers, using our main group-based specification with $gap > 0.25\%$. It also shows the corresponding unconditional ergodic average origination distribution in our Approximate Pooling MPE. For alternative specifications, Table 4.2 shows the MLE specification for $gap > 0\%$ and Table 4.3 for $gap > 0.5\%$.

Baseline: weighted by avg loan, $gap > 0.25\%$

χ	$p(\chi)$	$H(\chi)$	$G(\chi)$	Std Error p
0.0	0.0	0.3995	0.2345	
0.1098	0.0091	0.1857	0.1598	
0.2864	0.0236	0.203	0.2312	
0.6724	0.0545	0.1519	0.2328	
2.2083	0.1681	0.0598	0.1418	

Table 4.1: Group-based estimation of the attention distribution, assuming $N = 5$ homogeneous groups, focusing on households and months with $gap > 0.25\%$, weighted by average loan amount. The average attention rate is $\overline{\chi}_H = 31\%$.

Spec: weighted by avg loan, $gap > 0\%$

χ	$p(\chi)$	$H(\chi)$	$G(\chi)$	Std Error p
0.0	0.0	0.3711	0.2185	
0.0978	0.0081	0.1839	0.1593	
0.2444	0.0202	0.2206	0.2493	
0.5242	0.0427	0.1572	0.231	
1.411	0.1109	0.0671	0.1418	

Table 4.2: Group-based estimation of the attention distribution, assuming $N = 5$ homogeneous groups, focusing on households and months with $gap > 0\%$, weighted by average loan amount. The average attention rate is $\overline{\chi}_H = 25\%$.

Spec: weighted by avg loan, $gap > 0.5\%$

χ	$p(\chi)$	$H(\chi)$	$G(\chi)$	Std Error p
0.0	0.0	0.4061	0.2319	
0.1313	0.0109	0.2041	0.1743	
0.366	0.03	0.2015	0.2343	
0.9343	0.0749	0.1331	0.2158	
3.3916	0.2462	0.0553	0.1438	

Table 4.3: Group-based estimation of the attention distribution, assuming $N = 5$ homogeneous groups, focusing on households and months with $gap > 0.5\%$, weighted by average loan amount. The average attention rate is $\overline{\chi}_H = 41\%$.

5 Quantitative implications

5.1 Attentions rates and covariates

In this section, we study the degree of redistribution amongst households of different *observable* characteristics. To do this, we simply compute the cross-sectional correlation between observable characteristic X_i and household i's attention intensity $\chi_{\alpha(i)}$. We also look at a monotone transformation of this attention intensity — $1/(\nu + \chi_{\alpha(i)})$ — which is expressed in units of time. Some of the covariates are measured at the household level, and others are measured at the ZIP code level.[4] Table 5.1 summarizes these correlations. Our results suggests that the cross-subsidies we are documenting are regressive — in the sense that lower income, smaller mortgage, and younger households tend to be less attentive, and thus pay on average greater mortgage interest payments than higher income, larger mortgage and older households.

5.2 Pricing errors

In this section we evaluate via simulation the pricing errors that arise from Assumption 2 — i.e. the assumption that investors assume either (a) a constant, or (b) a state-dependent origination distribution when pricing newly-issued mortgages. In order to compute investor pricing errors, we perform the following computations:

1. We simulate $T = 100$ million consecutive months for the interest rate process r_t, starting with $r_0 = \mathbb{E}[r_t]$;

[4] Household ZIP code level covariate is the average of the household's time series over the relevant zip code value.

covariate X_i	$\mathbb{C}\text{orrel}\left(\chi_{\alpha(i)}, X_i\right)$	$\mathbb{C}\text{orrel}\left(\frac{1}{\nu+\chi_{\alpha(i)}}, X_i\right)$
avg. FICO	0.12	-0.27
avg. principal balance	0.12	-0.11
avg. CLTV	-0.11	0.15
less than high school education (zip)	-0.03	0.04
high school education (zip)	-0.01	0.00
some college education (zip)	0.01	-0.01
bachelor and above education (zip)	0.03	-0.03
median income (zip)	0.06	-0.07
median age (zip)	0.01	-0.02
population size (zip)	-0.04	0.09
white share (zip)	0.02	-0.05

Table 5.1: Correlation between attention and various covariates

2. Starting with an (arbitrary) distribution $f_0(c, \chi)$ over coupon and types[5], we compute, for our random path $\{r_t\}_{t\geq 0}$, the model-implied distribution f_t;

3. From f_t, we extract the origination distribution g_t as well as the prepayment flow mass $flow_t$.

4. Using the actual origination distribution G_t and the equilibrium mortgage rate $m(r_t, G)$ at time t, we derive the pricing error made by investors when originating mortgages at such time, i.e.

$$\mathbb{E}^{G_t}\left[P\left(r_t, m(r_t, G); \chi\right)\right] - (1+\pi)$$

5. By construction, the flow-weighted mean expected pricing error converges to zero, since

$$\lim_{T\to\infty}\sum_t^T \frac{1}{\sum_s^T \mathbb{1}_{\{s\in\{\tau:r_\tau=r\}\}}} w_t \times \mathbb{E}^{G_t}\left[P\left(r_t, m(r_t, G(\chi|r_t)); \chi\right)\right] = 1+\pi$$

where

$$w_t := \frac{\mathbb{1}_{\{t\in\{\tau:r_\tau=r\}\}} flow_t}{\sum_s^T \mathbb{1}_{\{s\in\{\tau:r_\tau=r\}\}} flow_s}$$

6. Instead, the flow-weighted standard deviation of pricing errors converges to a state-dependent non-zero constant, depicted in Figure 5.1.

[5] In practice, we use the ergodic conditional distribution $f_\infty(c, \chi|r_0)$.

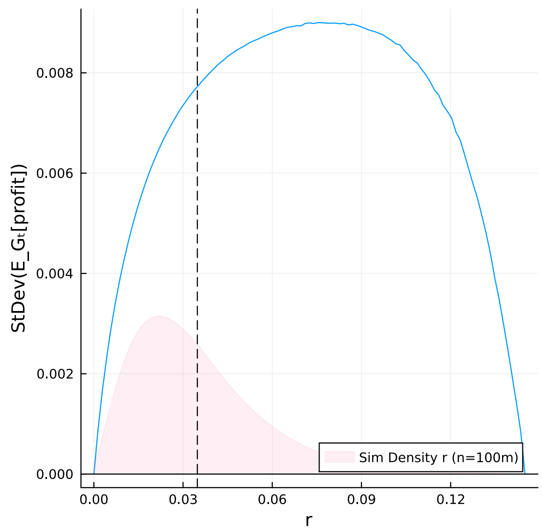

Figure 5.1: **Pricing errors:** Standard deviation (solid blue line) of pricing error conditional on r for a random path of r_t that is $T = 100m$ months long, under $m(r, G)$ pricing, where G is the conditional ergodic origination density. The pink density is the scaled number of observations that the process spends at each interest rate state.

The maximum conditional standard deviation is 90bps, and it is achieved at $r = 7.5\%$, while the standard deviation at the ergodic mean short rate is 77bps. The standard deviation vanishes at $r = \underline{r} = 0$: at this (lowest possible) value, as shown in to Proposition 8, the mortgage rate is invariant to the cross-sectional distribution. The standard deviation also vanishes at $r = \overline{r} = 14.5\%$, the upper bound on our state space, for similar reasons: at this level of interest rates, no household wants to strategically refinance, which means that the origination distribution must always be $G_t = H$. This standard deviation of pricing errors can be interpreted as mispricing risk as it arises out of the simplified pricing assumption. Thus, while on average investors break even, they are bearing some mispricing risk.

6 Small business credit market application

Consider a continuum of risk-neutral small firms of measure 1. Each firm generates income normalized to 1, has debt with notional balance normalized to b[6], and technology that fails with intensity $\lambda(x_t, \chi)$, with $\lambda(\cdot, \cdot)$ a known positive function that is increasing in x_t and increasing in χ. x_t is an observable, aggregate variable that represents the state of the economy; it follows a diffusion with drift $\mu(x)$ and volatility

[6] The parameter b can thus be interpreted as the debt-to-income ratio of a given firm.

coefficient $\sigma(x)$. χ is a firm-specific, time-invariant object that represents the intrinsic quality of the firm's project. Importantly, χ is not observable by the banking sector.[7] The firm quality distribution in the economy is H. When a firm's project fails, the firm ends up defaulting on its existing debt. At such time, a new firm immediately enters the economy, with the same quality, so as to preserve the distribution H over permanent quality heterogeneity.

The banking sector is risk-neutral and competitive; banks provide funding to small firms via loan contracts that mature at Poisson arrival rate ν and that carry an interest rate equal to the sum of (i) the risk-free rate r and (ii) a credit spread s_t, fixed and determined at the time the loan is originated, and meant to compensate banks for expected future credit losses. Firms have the option to refinance their bank debt early, subject to potential refinancing frictions—they must bear fixed debt issuance costs ψ, and only make decisions at discrete *points* in time, arriving with intensity α.

A firm currently financed with a loan at interest rate spread s has equity value

$$
\begin{aligned}
V(x,s) :&= \sup_{a\in\mathcal{A}} \mathbb{E}_{x,s}\left[\int_0^{\tau_\chi} e^{-rt}\left(1-\left(r+s_t^{(a)}\right)b\right)dt - a_t\psi dN_t^{(\alpha)}\right], \\
\text{s.t.} \qquad ds_t^{(a)} &= \left(\mathcal{S}(x_t)-s_{t-}^{(a)}\right)\left(a_t dN_t^{(\alpha)} + dN_t^{(\nu)}\right),
\end{aligned}
$$

where $\mathcal{A}$ is a set of progressively measurable binary actions $a=\{a_t\}_{t\geq 0}$ such that $a_t\in\{0,1\}$ at all times, $\mathcal{S}(x_t)$ is the equilibrium credit spread charged by banks on new loans when the aggregate state of the economy is x_t, τ_χ is the firm's default time (with time-varying intensity $\lambda(x_t,\chi)$), $N_t^{(\nu)}$ (resp. $N_t^{(\alpha)}$) is a counting process for maturity events (resp. refinancing decisions).

Firms refinance whenever the economy is improving "sufficiently"; their decision depends on the spread s over the risk-free rate currently paid on their loan. Specifically, a firm optimally refinances when $s-\mathcal{S}(x)\geq\theta(x)$, where the state-dependent spread threshold $\theta(\cdot)$ satisfies

$$
V(x,\mathcal{S}(x))-\psi = V(x,\mathcal{S}(x)+\theta(x))
$$

Banks are competitive when offering new loans to a new customer firm. The *shadow* price of a given \$1 notional loan to a borrower with known quality χ is given by

$$
P(x,s;\chi) = \mathbb{E}_x\left[\int_0^{\tau_\chi\wedge\tau_\theta} e^{-(r+\nu)t}(r+s+\nu)\,dt + \mathbb{1}_{\tau_\chi<\tau_\theta}e^{-(r+\nu)\tau_\chi}\rho + \mathbb{1}_{\tau_\chi>\tau_\theta}e^{-(r+\nu)\tau_\theta}\right],
$$

where τ_χ (resp. τ_θ) is the firm's default time (resp. loan prepayment time), and ρ is the recovery rate realized by creditors upon a default. Banks break-even when a

[7] This assumption can easily be relaxed, by assuming for example that investors can also rely upon a public and noisy signal of firm quality; in that case, the equilibrium would result in separation based on the public signal, and pooling for the private signal.

new loan is issued; they price loans under a (potentially state-dependent) firm quality distribution G, so that their no-profit loan origination condition can be written

$$\mathbb{E}^G \left[P(x, \mathcal{S}(x); \chi) \right] = 1$$

The origination distribution G is distinct from the distribution over firm quality H, with G skewed towards riskier firms, since (i) high risk firms will have a refinancing spread threshold θ that is lower than that of low risk firms, and (ii) riskier firms default at higher intensity, and are replaced by firms with identical quality that will immediately seek loan funding. In this model, low quality firms are subsidized by high quality firms, since they finance themselves at credit spreads more advantageous than if banks could observe the firm quality χ. An improvement in aggregate credit market conditions triggers a wave of loan refinancing events, consistent with the data. This model emphasizes the capital misallocation taking place in the banking sector due to the unobserved firm quality and the competitive banking sector.

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