Unions: Wage Floors, Seniority Rules, and Unemployment Duration*

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Abstract

This paper examines the impact of unions on unemployment and wages in a dynamic equilibrium search model. We model a union as imposing a minimum wage and rationing jobs to ensure that the union's most senior members are employed. This generates rest unemployment, where following a downturn in their labor market, unionized workers are willing to wait for jobs to reappear rather than search for a new labor market. We characterize the hazard rate of exiting unemployment, and show that it is low at long durations whenever the union-imposed minimum wage is high; we establish that a high union-imposed minimum wage generates a compressed wage distribution and a high turnover rate of jobs — properties consistent with the data. Finally, we show that seniority rules lead to lower unemployment levels, relative to an alternative rule allocating jobs to workers randomly.

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1 Introduction

This paper examines the impact of unions on unemployment and wages. We model a unionized labor market as one that imposes a minimum wage on employers and a seniority rule to allocate jobs among members. In particular, if the minimum wage binds, the union rations jobs to ensure that its most senior members are employed. Our focus is on the implications of such a policy on workers' decision to enter and exit unionized labor markets, on the duration of unemployment spells, on the unemployment rate and on wage compression. In our set-up, we abstract from any of the potentially beneficial roles of unions, and hence the allocation in a unionized labor market is inefficient.[1] We find that for the same union wage premium, the use of seniority to allocate jobs relative to using the commonly assumed random rule reduces unemployment and increases efficiency. We prove that, in the presence of search frictions, a laid-off union member will never immediately exit her labor market to search elsewhere for a job. Instead, she will endure a spell of "rest" unemployment, meaning that she will remain idle waiting for conditions to improve in her labor market and for more junior workers to exit. Thus, the hazard rate of reentering employment generally declines during an unemployment spell, so unionized workers will experience both frequent short spells and infrequent long spells of unemployment.

Our modeling strategy follows Alvarez and Shimer (2011), who build on Lucas and Prescott (1974). The economy consists of a large number of labor markets that produce imperfect substitutes. There are many workers and firms in each labor market, so in the absence of unions, wages and output prices are determined competitively within each labor market. Productivity shocks induce workers to move between labor markets. We begin by assuming that workers can move at no cost between labor markets.

Consider first the text-book model of unions. In the competitive sector, workers earn a wage w^*. In the unionized sector, they earn a higher wage $\hat{w}$ if employed, and zero otherwise; if there is excess demand for jobs, these jobs are randomly allocated to workers. In equilibrium, risk-neutral workers' indifference between competitive and unionized labor markets determines the unemployment rate u in the unionized sector:

[1] Our model fits into the "monopoly union" approach which stresses that unions may distort labor market outcomes by raising wages and rationing jobs, and abstracts from the potentially beneficial effects of unions, e.g. the "collective voice/institutional response" stressed by Freeman and Medoff (1984).

$u = 1 - w^*/\hat{w}$, increasing in the wage in the unionized sector relative to that in the competitive sector.

Consider then a unionized labor market with a seniority rule, according to which, when there is excess demand of unionized jobs, employment is allocated to the seniormost workers. We argue that this rule reduces unemployment in the unionized labor market. When unions use seniority to allocate jobs, not all workers are equally likely to be unemployed. The marginal worker — the worker with the lowest seniority — faces the worst employment prospects. Infra-marginal workers, i.e. those with higher seniority, are better off, either (i) ahead in the queue to get a job, or (ii) already employed. With discounting, since the marginal worker's payoff is back-loaded (relative to that of the worker in a labor market governed by a random allocation rule), her indifference between locating in a unionized or competitive labor markets must reduce the equilibrium unemployment rate, relative to the static textbook random assignment of jobs.

Our frictionless model is too stylized to address wage compression and properly distinguish between workers who enter or exit a unionized labor market. We thus turn to an extension of our model that includes search frictions. Similar to Alvarez and Shimer (2011), we distinguish between rest and search unemployment. While in rest unemployment, individuals do not work, enjoying a value of leisure higher than that of working but lower than being outside the labor force. Moreover, the rest unemployed retain the possibility of returning instantly and at no cost to the labor market where they last worked. Search unemployment enables a worker to locate in any labor market.

We show that if a union has any effect, it generates rest unemployment. Whenever the minimum wage $\hat{w}$ binds, workers with low seniority who are rationed out of a job decide to stay in the labor market, waiting for conditions to improve so that they can return to work at the minimum wage. If conditions improve immediately after a worker has been rationed out of a job, such worker re-enters employment quickly. When labor market conditions are bad enough, workers with the lowest seniority and out of a job will leave the unionized labor market and begin searching for work in more attractive markets. Thus, our model delivers both many short but also few long spells of unemployment; in general, the hazard rate of exiting unemployment is downward sloping.

The prospects of a labor market are limited by the fact that as conditions improve, new workers will arrive via search. These newcomers will have the lowest seniority, and hence will be most vulnerable to bad shocks, but they will only arrive in a labor market when it is booming. The situation of newcomers depends on how high the minimum wage is. If it is not too high, so that it binds only for bad shocks, they will immediately start working. If the minimum wage is sufficiently high, it always binds. In that case, newcomers arrive when prospects are very good, but are forced to queue until enough good shocks have arrived before they can start work. In such a labor market, there is always a queue of workers waiting either to start or resume employment. Thus, depending on the level of the union-mandated minimum wage, unionized labor markets can exhibit either no wage dispersion — and hence maximum wage compression — for very high minimum wage, or almost the same wage dispersion as competitive labor markets, if the minimum wage is set low enough to rarely bind.

Finally, we consider a union that sets the wage, or equivalently the employment level, at each instant in order to maximize the utilitarian welfare of all of its members, insiders and outsiders. We find that such union opts for a minimum wage policy, where the minimum wage is a constant markup over the marginal rate of substitution between the leisure value of rest unemployment and consumption. By setting this minimum wage, the union effectively restricts output so that it never exceeds the monopoly level. In our framework, the difference between a monopoly producer and a monopoly union is simply an issue of who keeps the monopoly rents.

Our paper is most similar to Alvarez and Shimer (2011), whose framework produces rest unemployment, arising from the leisure advantage of resting over working. We instead focus on the possibility that rest unemployment arises because of unionization and binding minimum wages. In Alvarez and Shimer (2011), all workers within a labor market are homogeneous, which means that the state, for a given worker, can be summarized by a single variable, function of productivity and the number of workers in such labor market. In this paper instead, union-mandated minimum wages and seniority rules imply that low seniority workers contribute less to the representative household's welfare than high seniority workers; thus, not only do we need to keep track of the labor market's condition, but also the worker's relative seniority, in order to understand their entry and exit decisions. From a technical standpoint, this means we have to solve a partial — rather than ordinary — differential equation to compute the worker's contribution to household welfare.

Our paper connects with the literature that examines the impact of unions on labor market outcomes. Using data on manufacturing firms at the state level, Medoff (1979) argues that unionized firms lay off workers at a much higher rate than non-unionized firms. Abraham and Medoff (1984) find evidence that seniority is an important determinant of layoffs in unionized firms. Blau and Kahn (1983) and Tracy (1986) suggest that seniority is also a factor in recall decisions of recently laid off or furloughed workers. More recently, Böckerman, Skedinger and Uusitalo (2018) study, using data from Finland and Sweden, how first-in-last-out rules affect layoff risks and wages, and find that such seniority rules reduce dismissal of older and more senior workers. Fujita and Moscarini (2017) document the importance of recalls of former employees after a jobless spell, and suggest that recalls are much more prevalent among union members, providing empirical support for the duration dependence of our model-implied hazard rate out of unemployment.

Jacobson, LaLonde and Sullivan (1993) document that workers displaced from heavily unionized industries suffer unusually large and persistent income declines, consistent with our model: in non-unionized labor markets, workers' welfare is limited by the possibility of new entrants coming to the labor market, whereas in unionized labor markets, high seniority workers are much better off than new entrants, leading to large welfare losses when these senior workers are displaced. Adamopoulou, Díez-Catalán and Villanueva (2022) investigate the impact of wage rigidities arising from collective bargaining agreements on labor market outcomes during recessions in Spain; they show that job losses in low-inflation recessions are entirely driven by workers with wages close to minimum floors set by collective bargaining agreements, echoing our theoretical conclusions that high minimum wages set by unions tend to increase average unemployment.

Our model also addresses a large literature arguing that unions compress wages. Blau and Kahn (1996) and Mourre (2005) document that wages in the U.S. are more dispersed than in other developed economies, and argue that this is due to the absence of centralized wage-setting mechanisms. Bertola and Rogerson (1997) show that such centralized wage setting mechanisms in Europe lead to wage compression whose effects on job creation and destruction rates is opposite that of restrictions on turnover; in our model, unions only affect labor market institutions by compressing wages, leading to a positive relationship between unemployment rates and wage compression.

Our approach to modeling unemployed union members as rest unemployed is related to that of Harris and Todaro (1970), who propose an extreme version of "wait unemployment" in less developed countries. When rural workers move to the city, they must queue for a job before they can start working. They are willing to do so even though the marginal product of labor is positive in the countryside. These findings are consistent with the version of our model with high — and always binding — minimum wage: unemployed individuals entering a new labor market have to wait for productivity to increase, and for their relative seniority to rise sufficiently, until they eventually reach the gates of the factory and get employed. Many of the forces driving the size of a labor market and unemployment in our framework — seniority rule, the ability for workers to "vote with their feet" and leave a labor market, entry whenever there is excess demand, indifference of the most junior worker between (i) the wage-and-unemployment probability package within a unionized labor market and (ii) employment opportunities elsewhere — can also be found in Grossman (1983), who studies a similar question but instead assumes that unions maximize the value of the *median* worker.

Although it is not our main focus, our paper gives a novel perspective on why unions might choose to raise wages above the market-clearing level. Many authors — see for instance Freeman and Medoff (1984) — have recognized that this may be optimal for more senior union members who are protected from the risk of layoff. Blanchard and Summers (1986) argue for an "insider-outsider" theory of European unemployment, where unions run by insiders generate unemployment because wages are set to exclude disenfranchised outsiders. We find that a union that cares equally about insiders and outsiders opts for a minimum wage policy: unions may generate unemployment not because more senior members may have an undue influence on wage setting procedures, but rather because they can only raise the well-being of all their members by constraining output in some states of the world. Finally, our model is consistent with the finding in Nickell and Layard (1999) that unions raise the unemployment rate only in countries where they cannot effectively coordinate their bargaining. In our model, the equilibrium without unions is Pareto optimal. While any individual union can improve its workers' well-being through a minimum wage, all workers are better off if unions do not exploit their monopoly power. Thus if unions can collude, they would be able to avoid generating rest unemployment.

2 Frictionless Model

We consider a continuous time, infinite-horizon model. We focus for simplicity on an aggregate steady state and assume markets are complete. Much of our analysis will focus on a specific labor market, which may or may not be unionized. A unionized labor market in our framework can be thought of as either (i) an industry whose workers are represented by a union — for instance, most of the workers of the largest three car manufacturers are unionized and represented by the United Auto Workers — or (ii) an occupation, and workers within that occupation are represented by a union — for example school teachers, who are represented by the American Federation of Teachers.

2.1 Goods

There is a continuum of goods indexed by $j \in [0,1]$ and a large number of competitive producers of each good. Each good is produced in a separate labor market with a constant returns to scale technology that uses only labor. In a typical labor market j at time t, there is a measure $\ell(j,t)$ of workers. Of these, $e(j,t)$ are employed, each producing $Ax(j,t)$ units of good j, while the remaining $\ell(j,t) - e(j,t)$ are rest-unemployed. Competition forces firms to price each good at marginal cost, so the wage in labor market j, $w(j,t)$, is equal to (a) the price of good j, $p(j,t)$, times (b) the productivity of each worker in labor market j, $Ax(j,t)$. A is the aggregate component in productivity while $x(j,t)$ is an idiosyncratic shock that follows a geometric Brownian motion,

$$d \ln x(j,t) = \mu_x dt + \sigma_x dz(j,t), \tag{1}$$

where μ_x measures the drift of log productivity, $\sigma_x > 0$ measures the volatility, and $z(j,t)$ is a standard Wiener process, independent across goods.

To keep a well-behaved distribution of labor productivity, the market for good j shuts down according to a Poisson process with arrival rate δ, independent across goods and independent of productivity. When this shock hits, all the workers are forced out of the labor market. A new good, also named j, enters with positive initial productivity $x \sim F(x)$, keeping the measure of goods constant. We assume a law of large numbers, so the share of labor markets experiencing any particular sequence of shocks is deterministic.

2.2 Households

There is a representative household consisting of a measure 1 of members. The large household structure allows for full risk sharing within each household, a standard device for studying complete markets allocations. At each time t, the representative household allocates each of her members to one of the following mutually exclusive activities:

- $L(t)$ household members are located in one of the labor markets.
  - $E(t)$ of these workers are employed at the prevailing wage and get leisure 0.
  - $U_r(t) = L(t) - E(t)$ of these workers are rest-unemployed and get leisure b_r.
- The remaining $1 - E(t) - U_r(t)$ household members are inactive, getting leisure b_i.

We assume $b_r < b_i$, so rest unemployment gives less leisure than inactivity. Household members may costlessly move between these three states. However, whenever they enter (or reenter) a market, they start with the lowest level of seniority. In addition to the endogenous decision to leave a market, we allow for two other exogenous reasons why a worker might exit her labor market. First, a given labor market shuts down at rate δ. Second, the worker might be hit by an idiosyncratic shock, which occurs according to a Poisson process with arrival rate q, independent across individuals and independent of their labor market's productivity. We introduce the idiosyncratic "quit" shock q to account for separations that are unrelated to the state of the labor market. We represent the household's preferences via the utility function

$$\int_0^\infty e^{-\rho t} \left(\ln C(t) + b_i\big(1 - E(t) - U_r(t)\big) + b_r U_r(t)\right) dt, \tag{2}$$

with $\rho > 0$ the discount rate and $C(t)$ the household's consumption of a composite good

$$C(t) = \left(\int_0^1 c(j,t)^{\frac{\theta-1}{\theta}} dj\right)^{\frac{\theta}{\theta-1}}, \tag{3}$$

and $c(j,t)$ the consumption of good j at time t. We assume that the elasticity of substitution between goods, θ, is greater than 1. The cost of this consumption is $\int_0^1 p(j,t)c(j,t)dj$, which we assume the household finances using its labor income. Standard arguments imply that the demand for good j satisfies

$$c(j,t) = \frac{C(t)P(t)^{\theta}}{p(j,t)^{\theta}}, \tag{4}$$

where

$$P(t) = \left(\int_0^1 p(j,t)^{1-\theta} dj\right)^{\frac{1}{1-\theta}} \tag{5}$$

is the price index, normalized to 1. To ensure a well-behaved distribution of wages, we impose two restrictions on preferences and technology. First, we require

$$\delta > (\theta - 1)\left(\mu_x + \tfrac{1}{2}(\theta-1)\sigma_x^2\right), \tag{6}$$

so labor markets shut down sufficiently quickly to offset the drift in the stochastic process for productivity; this is a necessary and sufficient condition to guarantee that aggregate consumption is strictly positive and finite. Second, we require

$$X \equiv \left(\int_0^\infty x^{\theta-1} dF(x)\right)^{\frac{1}{\theta-1}} \in (0,\infty), \tag{7}$$

a restriction on the productivity distribution in new labor markets. If this condition failed, the wage would be either zero or infinite.

2.3 Unions

Unions constrain the wage in labor market j, introducing a restriction $w(j,t) \geq \hat{w}(j)$. For most of our analysis, we treat the minimum wage $\hat{w}(j)$ as exogenous and consider its consequences. To see whether the minimum wage constraint binds, first note that if all the workers in the labor market were employed, they would produce $Ax(j,t)\ell(j,t)$ units of good j. Inverting the demand curve (4) and eliminating the price level using $P(t) = 1$, the relative price of good j would be

$$p(j,t) = \left(\frac{C(t)}{Ax(j,t)\ell(j,t)}\right)^{\frac{1}{\theta}}.$$

The wage in the labor market would then be $p(j,t)Ax(j,t)$ or

$$w(j,t) = \left(\frac{C(t)\big(Ax(j,t)\big)^{\theta-1}}{\ell(j,t)} \right)^{\frac{1}{\theta}}, \tag{8}$$

increasing in productivity and decreasing in the number of workers. If there are too many workers in such labor market, the minimum wage constraint binds. In that case, $w(j,t) = \hat{w}(j)$ and employment $e(j,t)$ is determined at the level that makes the price of good j equal to $\hat{w}(j)/Ax(j,t)$,

$$e(j,t) = \frac{C(t)\big(Ax(j,t)\big)^{\theta-1}}{\hat{w}(j)^{\theta}}, \tag{9}$$

increasing in productivity and decreasing in the minimum wage.

We assume that when the minimum wage constraint binds, more senior workers have the first option to work, where seniority is measured by the elapsed time since the worker last entered the labor market. Consider a worker with relative seniority $s \in [0,1]$, where we measure relative seniority s as the percentage of workers in the labor market with lower seniority, so $s = 1$ corresponds to the worker with the greatest seniority. She is employed if and only if $e(j,t)/\ell(j,t) \geq 1 - s$ or, from (9),

$$s \geq 1 - \frac{C(t)\big(Ax(j,t)\big)^{\theta-1}}{\hat{w}(j)^{\theta}\ell(j,t)}. \tag{10}$$

A worker with a given seniority is more likely to be employed when productivity is higher, the minimum wage is lower, or the number of workers in the labor market is smaller.

Since workers are typically not indifferent about working, those with more seniority provide more utility to the representative household. Thus to analyze a worker's decision to enter or stay in a labor market, we need to examine not only the behavior of wages in the market, but also how the entry and exit of other workers influences each worker's seniority.

2.4 Equilibrium

We look for a competitive equilibrium of this economy, subject to the constraints imposed by minimum wages. At each instant, each household chooses how much of each good to consume and how to allocate its members between employment, rest unemployment, and inactivity, in order to maximize utility subject to the constraints imposed by seniority rules; and each goods producer j maximizes profits by choosing how many workers to hire taking as given the wage in its labor market and the price of its good. The demand for labor from goods producers is equal to the supply from households in each market unless the minimum wage constraint binds, in which case labor demand may be less than labor supply; and households' demand for goods is equal to the supply from firms. We focus on parameter values for which the household keeps some of its members inactive, which requires that the leisure value of inactivity b_i is sufficiently large.

We look for a stationary equilibrium where all aggregate quantities and prices are constant, as is the joint distribution of wages, productivity, output, employment, and rest unemployment across labor markets. We suppress the time argument as appropriate in what follows. With identical households and complete markets, consumption is equal to current labor income and hence we also ignore financial markets.

2.5 Characterization

In this section, we prove that the number of workers in labor market j satisfies

$$\ell(j,t) = \frac{C\big(Ax(j,t)\big)^{\theta-1}}{\bar{w}(j)^{\theta}} \tag{11}$$

for some constant $\bar{w}(j)$, where C is the constant level of consumption. We also characterize $\bar{w}(j)$. In unionized markets with a binding minimum wage $\hat{w}(j)$, we prove that $\bar{w}(j) < \hat{w}(j)$. Equation (10) then implies that a worker is employed if and only if

$$s \geq 1 - \left(\frac{\bar{w}(j)}{\hat{w}(j)}\right)^{\theta} \equiv \hat{s}(j) \in (0,1). \tag{12}$$

The unemployment rate in labor market j is equal to $\hat{s}(j)$. In labor markets where the minimum wage $\hat{w}(j)$ is not binding, $\bar{w}(j) = w^*$, a constant that satisfies $w^* \geq \hat{w}(j)$. All workers are employed and have the same expected utility, regardless of their seniority. In what follows, we suppress the name of the labor market j.

2.5.1 Non-unionized labor market

First, consider a non-unionized labor market. Regardless of the sequence of shocks hitting the labor market, a worker must earn a constant wage w^* and is always employed. Indeed, we have assumed an equilibrium in which some members of the household are inactive. Since the household can freely move workers between inactivity and a job in a non-unionized labor market, it must be indifferent between the two activities. An inactive worker contributes b_i utils to the household, while a worker employed at w^* contributes w^*/C, since the marginal utility of consumption is $1/C$. Thus, we must have $w^* = b_i C$. As long as the minimum wage is smaller than this level, $\hat{w} \leq w^*$, it does not bind. As a labor market with a non-binding minimum wage is hit by productivity shocks, the number of workers varies according to (11), while the wage stays constant at w^*. The workers in such labor markets move between inactivity and employment as necessary while avoiding any unemployment spells.

2.5.2 Unionized labor market

Now consider the case where $\hat{w} > w^* = b_i C$. The analysis in the previous paragraph is inapplicable because the minimum wage is binding. We conjecture that in equilibrium a worker's value in a labor market with a binding minimum wage depends only on her relative seniority $v(s)$, where $s \in [0, 1]$ is the fraction of workers with lower seniority. A worker exits a labor market (i) at the time $\tau(0)$ when her seniority $s(t)$ falls to 0 and the labor market is hit by an adverse shock, (ii) at the time τ_q she is hit by an exogenous quit shock, or (iii) at the time τ_δ her labor market shuts down. She works and earns the minimum wage $\hat{w}$ whenever her seniority exceeds the threshold defined in (12) for some value of $\bar{w} < \hat{w}$ to be determined; whenever the worker's seniority is below the threshold, she is rest unemployed, with flow utility b_r. In other words, the per-period worker's flow payoff is $R(s)$, defined via

$$R(s) = \begin{cases} b_r & \text{if } s < \hat{s} \\ \hat{w}/C & \text{if } s \geq \hat{s}. \end{cases} \tag{13}$$

The value function $v(s)$ can then be expressed via:

$$v(s) = \mathbb{E}_s \left[\int_0^{\tau(0)\wedge\tau_\delta\wedge\tau_q} e^{-\rho t} R\left(s(t)\right) dt + e^{-\rho\left(\tau(0)\wedge\tau_\delta\wedge\tau_q\right)} \frac{b_i}{\rho} \right], \tag{14}$$

where expectations are taken with respect to the stopping times $\tau(0)$, τ_δ, τ_q, and the path of the state $s(t)$ prior to the stopping time, where $\wedge$ is the minimum operator, and where the subscript $\mathbb{E}_s$ means that we condition on the initial state being $s(0) = s$.

Whenever a productivity shock $d \ln x(t)$ occurs, the measure of employed workers $e(t)$ adjusts according to (9), such that $d \ln e(t) = (\theta - 1) d \ln x(t)$. Since we postulated a constant employment rate within such labor market, $e(t)/\ell(t)$ must stay constant, implying that $d \ln \ell(t) = (\theta - 1) d \ln x(t)$, in other words workers leave such labor market (if $d \ln x(t) < 0$) or arrive (if $d \ln x(t) > 0$). These comments allow us to determine the dynamics of the seniority level $s(t)$ of a particular individual in a particular labor market at time t. Supposed $s(t) \in (0, 1)$. This means that there is a measure $s(t)\ell(t)$ of workers with seniority lower than the seniority of that particular individual in such labor market. Let us assume that between t and $t + dt$, productivity shocks lead to a change $d\ell(t)$ in the measure of workers in this particular labor market. Conditioning on the labor market not shutting down and on the worker not being hit by a quit shock, the change in the measure of workers junior to the particular individual must then verify:

$$d\left(s(t)\ell(t)\right) = d\ell(t) + q\ell(t)\left(1 - s(t)\right) dt \tag{15}$$

The first term on the right-hand side corresponds to the change in workers junior to a given worker arising from endogenous entry and exits following productivity shocks, while the second term relates to exogenous quits. Since in our assumed equilibrium, $d \ln \ell(t) = (\theta - 1) d \ln x(t)$, using Itô's lemma (see Appendix A.1.1) enables us to identify the dynamics of the seniority of a particular individual:

$$ds(t) = (1 - s(t)) \left[q + (\theta - 1) \left(\mu_x - \frac{1}{2}(\theta - 1)\sigma_x^2 \right) \right] dt + (1 - s(t))(\theta - 1)\sigma_x dz(t) \quad (16)$$

Thus, $1 - s(t)$ follows geometric Brownian motion dynamics with log-normal drift rate $-q - (\theta - 1) \left(\mu_x - \frac{1}{2}(\theta - 1)\sigma_x^2 \right)$ and log-normal volatility $(\theta - 1)\sigma_x$.

Consider then the boundary behavior of v. Since workers with zero seniority are free to leave a unionized labor market to join a competitive labor market, $v(0)$ must be equal to the reservation value b_i/ρ. Since the timing of labor market entry and exit is chosen optimally (see Appendix A.1.4), we must have the smooth pasting condition $v'(0) = 0$. Similarly, Brownian shocks $dz(t)$ smooth the value function at the threshold $s = \hat{s}$, and lastly $s = 1$ is an absorbing state, which allows us to determine the value of the senior-most worker. Together with the state dynamics, these boundary conditions allow us to solve for the value function v analytically (see Appendix A.1.3):

Proposition 1. Let $\lambda \equiv q + \delta$ be the exogenous rate at which workers exit labor markets. Let $\eta_1 < 0 < 1 < \eta_2$ be the roots of the characteristic equation

$$\frac{1}{2}(\theta - 1)^2\sigma_x^2\eta^2 + (q + (\theta - 1)\mu_x)\,\eta - (\rho + \lambda) = 0. \quad (17)$$

Equation (A3) in Appendix A.1.3 gives a closed form solution for the worker's value function. In equilibrium, the unemployment rate in a given market is constant and equal to

$$\hat{s} = 1 - \left(\frac{w^* - b_r C}{\hat{w} - b_r C} \right)^{\frac{1}{\eta_2}} = 1 - \left(\frac{\bar{w}}{\hat{w}} \right)^{\theta}. \quad (18)$$

The number of workers $\ell(t)$ varies with productivity according to Equation (11), where

$$\bar{w} = \hat{w} \left(\frac{w^* - b_r C}{\hat{w} - b_r C} \right)^{\frac{1}{\theta\eta_2}} < \hat{w}. \quad (19)$$

2.6 Unemployment

Equation (18) describes the unemployment rate $\hat{s}$ in a labor market with minimum wage $\hat{w} \geq w^*$. It is equal to 0 if $\hat{w} \leq w^*$ and is then increasing in the minimum wage $\hat{w}$. To understand the magnitude of unemployment, compare this to a hypothetical labor market with a minimum wage but where jobs are allocated randomly, not based on seniority. If a worker enters such a labor market, she is employed at the minimum wage $\hat{w}$ with probability $1-u$ and rest-unemployed otherwise. Since a household must be indifferent between sending a worker to such a labor market and sending the worker to a competitive labor market, we have

$$w^*/C = (1-u)\hat{w}/C + ub_r \Rightarrow u = 1 - \frac{w^* - b_r C}{\hat{w} - b_r C}.$$

Since $\hat{w} > w^* = b_i C > b_r C$, this defines $u \in (0,1)$. And since $\eta_2 > 1$, we can immediately conclude with the following corollary.

COROLLARY 1. The unemployment rate with seniority rule is lower than with a random allocation of jobs to workers, $u > \hat{s}$.

Relative to a random assignment rule, the seniority rule backloads the worker's payoff into the future. With discounting — either stemming from the impatience parameter ρ, the quit rate q, or the labor market shut down rate δ — this backloading lowers the utility of newcomers in a labor market, so long as the incentive to queue is not too strong. To preserve the indifference for a newcomer between inactivity and queuing into a new labor market, the equilibrium unemployment rate must be lower.[2]

With a random assignment of jobs to union members, the unemployment rate depends only on the leisure from inactivity and rest unemployment and the real wage

[2] Our argument depends on the strength of the incentive to queue in a labor market with seniority rule, relative to that in a labor market with random allocation of jobs to workers. That incentive depends on how the seniority rule affects future employment in the labor market. In the limit case without discounting, $\rho = 0$, if the number of workers $\ell(t)$ and employment $e(t)$ in a labor market are martingales, the value for a newcomer in a labor market with seniority rule is identical to their value if jobs are assigned randomly, so the unemployment rates in the seniority rule and random allocation models must be identical. Condition (6) insures that the number of workers and employment in a labor market are supermartingales. This means that work is backloaded with seniority rules, and so the incentive to queue is lower than in an economy with a random allocation of jobs to workers, even without any discounting.

$\hat{w}/C$. With seniority rule, other preference and technology parameters also affect a labor market's unemployment rate through their effect on η_2; (18) implies that any parameter which raises η_2 reduces the unemployment rate.[3]

Proposition 2. The unemployment rate $\hat{s}$ is decreasing in (a) the discount rate ρ and (b) the labor market shut-down rate δ. It is increasing in (a) the quit rate q, (b) the productivity drift μ_x, and (c) the elasticity of substitution θ. Keeping the log-normal drift rate of the number of workers $\ell(j,t)$ in labor market j constant, an increase in productivity volatility σ_x increases the unemployment rate.

Greater impatience ρ or a higher labor market shut-down rate δ leads to more effective discounting, which raises η_2 and hence reduces the unemployment rate. The intuition for this comparative static is similar to that discussed in Corollary 1: since marginal workers are always unemployed, an increase in effective discounting implies that workers weigh current unemployment more heavily than the future possibility of employment and so are less inclined to stay in the labor market.

A higher quit rate q has two offsetting effects on the unemployment rate. On the one hand, a higher q leads to greater effective discounting, which pushes unemployment rate lower; on the other hand, a higher q means that a worker's seniority grows more rapidly, making such labor market more attractive, thereby raising unemployment rate; under condition (6), this latter force prevails. Similarly, higher productivity drift μ_x raises the unemployment rate, since it also increases the attractiveness of the labor market. A greater elasticity of substitution θ raises the unemployment rate because it amplifies the impact of any productivity shock. Finally, keeping the drift rate of the number of workers in labor market j constant means that we fix $\mathbb{E}_t\left[d\ell(j,t)\right] = \left[(\theta-1)\mu_x + \frac{1}{2}(\theta-1)^2\sigma_x^2\right]dt$. In such case, an increase in productivity volatility increases a worker's option value of waiting to see how productivity evolves, thus raising unemployment. None of these possibilities are present in the static model.

Finally, we note that we can easily "close" our model by leveraging equations (11)

[3]The discussion in this paragraph and the next two paragraphs is loose because we implicitly assume that a change in parameters does not affect the level of consumption C. In the next section, we extend the model to have many labor markets and allow these parameters to differ across markets. If we followed a similar approach here, the comparative statics with respect to λ, μ_x, σ_x, and θ would be relevant in the cross-section. Alternatively, one can assume $b_r = 0$, in which case our comparative static results do not depend on equilibrium aggregate consumption C.

and (19) to obtain an expression for $c(j,t)$ as a function of the minimum wage and of $w^* = b_i C$, and then use the household's composite consumption bundle (3) to obtain an implicit equation that C needs to satisfy in equilibrium. We provide those detailed calculations in Appendix A.1.5.

3 Full Model

We now extend the model by introducing search frictions. While workers can costlessly move between employment and rest unemployment within a labor market, we assume it takes time to move between markets. This changes our results along several dimensions.

First, productivity shocks cause wage fluctuations within labor markets since search frictions prevent costless arbitrage of any wage differences across markets. With wage fluctuations, we interpret unions as imposing a minimum wage $\hat{w}$ and a seniority rule, rather than just a fixed wage. Following a positive sequence of productivity shocks, the minimum wage constraint may be slack and all the union members employed. More generally, in the presence of search frictions some markets may be more attractive than others, even for a worker without seniority.

Second, workers need not experience a spell of unemployment when they enter a market. Workers enter markets with a moderate minimum wage at times when the minimum wage constraint does not bind. This allows them to start a job immediately. But when their market is hit by an adverse shock, they will not immediately exit. Instead, we prove that they will always experience a spell of rest unemployment before exiting. In this sense, rest unemployment is associated with declining unionized labor markets. Still, for a sufficiently high minimum wage relative to the search frictions, the minimum wage will always bind and so the market will always have some unemployment.

Finally, search frictions lead to the notion of workers "attached" to a labor market. This allows us to consider the objective function of a union that represents those workers.

We also extend the model along another dimension. We assume there are many sectors that produce relatively poor substitutes. Within each sector, there are many labor markets producing goods that are relatively easily substituted. This facilitates comparative statics at the cost of somewhat more cumbersome notation.

3.1 Goods

There is a continuum of sectors indexed by $n \in [0,1]$. Within each sector, there is a continuum of goods indexed by $j \in [0,1]$ and a large number of competitive producers of each good. Thus n_j is the name of a particular good produced in a particular labor market. The model from the previous section applies within each sector, although parameters may differ across sectors. In labor market n_j at time t, there is a measure $e(n_j, t)$ employed workers, each of whom produce $Ax(n_j, t)$ units of good n_j. There are also $\ell(n_j, t) - e(n_j, t)$ rest-unemployed workers. Workers are paid their marginal product, so the wage in market n_j solves $w(n_j, t) = p(n_j, t)Ax(n_j, t)$, where $p(n_j, t)$ is the price of good n_j.

A is the aggregate component in productivity while $x(n_j, t)$ is an idiosyncratic shock that follows a geometric random walk with sector-specific drift $\mu_{n,x}$ and sector-specific standard deviation $\sigma_{n,x}$:

$$d \ln x(n_j, t) = \mu_{n,x} dt + \sigma_{n,x} dz(n_j, t). \tag{20}$$

As before, the market for good n_j shuts down at Poisson times with arrival rate δ_n, independent across goods and productivity. When this shock hits, all the workers are forced out of the labor market. A new good, also named n_j, enters with positive initial productivity $x \sim F_n(x)$, keeping the measure of goods in sector n constant. We assume a law of large numbers, so the share of labor markets in each sector experiencing any particular sequence of shocks is deterministic.

3.2 Households

There is a representative household consisting of a measure 1 of members. At each moment in time t, the representative household allocates each of her members to one of the following mutually exclusive activities:

- $L(t)$ household members are located in one of the labor markets.
  - $E(t)$ of these workers are employed at the prevailing wage and get leisure 0.
  - $U_r(t) = L(t) - E(t)$ of these workers are rest-unemployed and get leisure b_r.

- $U_s(t)$ household members are search-unemployed, looking for a new labor market and getting leisure b_s.
- The remaining $1 - E(t) - U_r(t) - U_s(t)$ household members are inactive, getting leisure b_i.

We assume $b_i > b_s$ but no longer impose $b_i > b_r$. Household members may costlessly switch between employment and rest unemployment and between inactivity and searching; however, they cannot switch labor markets without going through a spell of search unemployment. Workers exit their labor market for inactivity or search in three circumstances: first, they may do so endogenously at any time at no cost; second, they must do so when their market shuts down, which happens at rate δ_n; and third, they must do so when they are hit by an idiosyncratic shock, according to a Poisson process with arrival rate q_n, independent across individuals and independent of their labor market's productivity. Finally, a worker in search unemployment finds a job according to a Poisson process with arrival rate α. When this happens, she may enter the labor market of her choice. We represent the household's preferences via the utility function

$$\int_0^\infty e^{-\rho t} \left(\ln \bar{C}(t) + b_i\big(1 - E(t) - U_r(t) - U_s(t)\big) + b_r U_r(t) + b_s U_s(t)\right) dt, \tag{21}$$

where $\rho > 0$ is the discount rate and $\bar{C}(t)$ is the household's consumption of an aggregate of all goods produced across sectors,

$$\ln \bar{C}(t) = \int_0^1 \ln C(n,t) dn, \tag{22}$$

$C(n,t)$ is the household's consumption of an aggregate of goods produced in sector n,

$$C(n,t) = \left(\int_0^1 c(n_j,t)^{\frac{\theta_n - 1}{\theta_n}} dj\right)^{\frac{\theta_n}{\theta_n - 1}}, \tag{23}$$

and $c(n_j,t)$ is the consumption of good n_j at time t. We assume that the elasticity of substitution between goods in sector n, θ_n, is greater than 1. The cost of this consumption is $\int_0^1 \int_0^1 p(n_j,t)c(n_j,t)djdn$, which we assume the household finances using its labor income.

Standard arguments imply that the demand for good n_j satisfies

$$c(n_j, t) = \frac{C(n, t) P(n, t)^{\theta_n}}{p(n_j, t)^{\theta_n}}, \tag{24}$$

where

$$P(n, t) = \left(\int_0^1 p(n_j, t)^{1-\theta_n} dj \right)^{\frac{1}{1-\theta_n}} \tag{25}$$

is the price index in sector n. The demand for the consumption aggregator in sector n satisfies

$$C(n, t) = \frac{\bar{C}(t)}{P(n, t)}, \tag{26}$$

where we use the price of the aggregate consumption bundle $\bar{C}$ as numeraire, or equivalently normalize

$$\int_0^1 \ln P(n, t) dn = 0. \tag{27}$$

To ensure a well-behaved distribution of wages in each labor market, we impose two restrictions on preferences and technology, generalizations of equations (6) and (7):

$$\delta_n > (\theta_n - 1)\big(\mu_{n,x} + \frac{1}{2}(\theta_n - 1)(\sigma_{n,x})^2\big) \tag{28}$$

$$X_n \equiv \left(\int_0^\infty x^{\theta_n - 1} dF_n(x) \right)^{\frac{1}{\theta_n - 1}} \in (0, \infty). \tag{29}$$

3.3 Unions

Unions constrain the wage in sector n, introducing a restriction $w(n_j, t) \geq \hat{w}(n)$. To see whether the minimum wage constraint binds, first note that if all the workers in the labor market were employed, they would produce $Ax(n_j, t)\ell(n_j, t)$ units of good n_j. Inverting the demand curve (24) and eliminating the price of labor market n using (26), the relative price of good n_j would be

$$p(n_j, t) = \frac{\bar{C}(t)}{C(n, t)^{\frac{\theta_n - 1}{\theta_n}} \big(Ax(n_j, t)\ell(n_j, t)\big)^{1/\theta_n}}.$$

The wage in the labor market would then be $p(n_j, t)Ax(n_j, t)$ or

$$w(n_j, t) = \frac{\bar{C}(t)\big(Ax(n_j, t)\big)^{\frac{\theta_n - 1}{\theta_n}}}{C(n, t)^{\frac{\theta_n - 1}{\theta_n}} \ell(n_j, t)^{\frac{1}{\theta_n}}}. \tag{30}$$

This is increasing in the productivity of the labor market and decreasing in the number of workers. In particular, if there are too many workers in the market, the minimum wage constraint binds. In that case, $w(n_j, t) = \hat{w}(n)$ and employment is determined at the level that makes the price of good n_j equal to $\hat{w}(n)/Ax(n_j, t)$,

$$e(n_j, t) = \frac{\bar{C}(t)^{\theta_n}\big(Ax(n_j, t)\big)^{\theta_n - 1}}{C(n, t)^{\theta_n - 1}\hat{w}(n)^{\theta_n}}, \tag{31}$$

increasing in productivity and decreasing in the minimum wage. We continue to assume that when the minimum wage constraint binds, more senior workers have the first option to work. When the minimum wage binds, a worker with relative seniority s works if and only if

$$s \geq 1 - \frac{\bar{C}(t)^{\theta_n}\big(Ax(n_j, t)\big)^{\theta_n - 1}}{C(n, t)^{\theta_n - 1}\hat{w}(n)^{\theta_n}\ell(n_j, t)}. \tag{32}$$

3.4 Equilibrium

A competitive equilibrium of this economy is defined similarly to that of the previous section. At each instant, each household chooses how much of each good to consume and how to allocate its members across different activities (employment, rest unemployment, search unemployment, and inactivity) in order to maximize utility subject to technological constraints, taking as given the behavior of wages and seniority in each labor market; and each goods producer n_j maximizes profits by choosing how many workers to hire taking as given the wage in its labor market and the price of its good. Demand for labor from goods producers is equal to the supply from households in each market unless the minimum wage constraint binds, in which case labor demand may be less than labor supply; and households' demand for goods is equal to the supply from firms. We focus on a stationary equilibrium where all aggregate and sector-specific quantities and prices are constant, as is the joint distribution of wages, productivity, output, employment, and rest unemployment across labor markets. We continue to assume complete markets.

4 Characterization of Equilibrium

At any point in time, a typical labor market n_j is characterized by its productivity x and the number of workers ℓ. We look for an equilibrium in which the ratio $x^{\theta_n - 1}/\ell$ follows a Markov process. Workers enter labor markets when the ratio exceeds a threshold and exit labor markets when it falls below a strictly smaller threshold. Moreover, (32) shows that this ratio and a worker's seniority determines whether she has the option to work.

4.1 The Marginal Value of Household Members

We start by computing the marginal value of an additional household member engaged in each of the three activities. These are related by the possibility of reallocating household members between activities. Consider first a household member who is permanently inactive. It is immediate from (21) that she contributes

$$\underline{v} = \frac{b_i}{\rho} \tag{33}$$

to household utility. Since the household may freely shift workers between inactivity and search unemployment, this must also be the incremental value of a searcher, assuming some members are engaged in each activity. A searcher gets flow utility b_s and the possibility of finding a labor market at rate α, giving capital gain $\bar{v} - \underline{v}$, where $\bar{v}$ is the value to the household of having a worker enter the best labor market. This implies $\rho \underline{v} = b_s + \alpha \left(\bar{v} - \underline{v} \right)$ or

$$\bar{v} = \underline{v} + b_i \kappa, \text{ where } \kappa \equiv \frac{b_i - b_s}{b_i \alpha} \tag{34}$$

is a measure of search costs, the percentage loss in current utility from searching rather than inactivity times the expected duration of search unemployment $1/\alpha$. Conversely, a worker may freely exit her labor market, and so the lower bound on the value of a household member in a labor market, either employed or search unemployed, is $\underline{v}$. If the household values a worker at some intermediate amount, it will be willing to keep her in her labor market rather than having her search for a new one.

Finally, consider the margin between employment and resting for a worker in a labor market paying a wage w. A resting worker generates b_r utils while an employed worker generates income valued at $w/\bar{C}$, where $1/\bar{C}$ is the marginal utility of the consumption aggregate. Since switching between employment and resting is costless, all workers prefer to work in any labor market with $w/\bar{C} > b_r$ and prefer to rest in any market with $w/\bar{C} < b_r$. This implies that if $\hat{w}/\bar{C} \le b_r$, the minimum wage never binds because workers' willingness to enter rest unemployment endogenously keeps the wage above $\hat{w}$. Conversely, if $\hat{w}/\bar{C} > b_r$, the minimum wage may sometimes bind.

4.2 Wage and Labor Force Dynamics

Consider a labor market n with ℓ workers, productivity x, and a minimum wage $\hat{w}$. Let $P(\ell, x)$ be the price of its good, $Q(\ell, x)$ the amount of the good produced, $W(\ell, x)$ the wage rate, and $E(\ell, x)$ the number of workers who are employed. Competition ensures that the wage is equal to the marginal product of labor, $W(\ell, x) = P(\ell, x)Ax$, while the production function implies $Q(\ell, x) = E(\ell, x)Ax$. From (30), the wage solves

$$W(\ell, x) = \bar{C} \max\{e^{\hat{\omega}}, e^{\omega}\} \tag{35}$$

where

$$\omega \equiv \frac{(\theta_n - 1)(\ln(Ax) - \ln C(n)) - \ln \ell}{\theta_n}, \tag{36}$$

is the logarithm of the "full-employment wage" measured in utils, the wage that would prevail if there were full employment in the labor market and

$$\hat{\omega} \equiv \max\{\ln \hat{w} - \ln \bar{C}, \ln b_r\} \tag{37}$$

is the maximum of the log minimum wage (in utils) and the utility from rest unemployment. From (31), employment is $E(\ell, x) = \ell e^{\theta_n(\omega - \hat{\omega})}$ if the minimum wage binds, $\omega < \hat{\omega}$, and ℓ otherwise. Hence the amount of the good produced is

$$Q(\ell, x) = \ell A x \min\{1, e^{\theta_n(\omega - \hat{\omega})}\}. \tag{38}$$

When $\omega \geq \hat{\omega}$, the wage exceeds the minimum wage and so there is no rest unemployment. Otherwise, enough workers rest to raise the log wage in utils to $\hat{\omega}$.

Since the wage only depends on ω, we look for an equilibrium in which any labor market with $\omega > \bar{\omega}_n(\hat{\omega})$ immediately attracts new entrants to push the log full employment wage back to $\bar{\omega}_n(\hat{\omega})$ and workers with the least seniority immediately exit any labor market with $\omega < \underline{\omega}_n(\hat{\omega})$ until the log full employment wage increases to $\underline{\omega}_n(\hat{\omega})$. The thresholds $\underline{\omega}_n(\hat{\omega}) \leq \bar{\omega}_n(\hat{\omega})$ are endogenous and depend on both labor market n and minimum wage $\hat{\omega}$. Workers neither enter nor endogenously exit from labor markets with $\omega \in (\underline{\omega}_n(\hat{\omega}), \bar{\omega}_n(\hat{\omega}))$, although a fraction of the workers $q_n dt$ quit during an interval of time dt. We allow for the possibility that $\underline{\omega}_n(\hat{\omega}) = -\infty$ so workers never exit labor markets. When a positive shock hits a labor market with $\omega = \bar{\omega}_n(\hat{\omega})$, ω stays constant and the labor force ℓ increases endogenously. When $\underline{\omega}_n(\hat{\omega}) < \omega < \bar{\omega}_n(\hat{\omega})$, both positive and negative shocks affect ω, while ℓ falls deterministically at rate q_n. When $\omega = \underline{\omega}_n(\hat{\omega})$, a negative shock reduces ℓ endogenously without affecting ω.

If there is an equilibrium with this property, its definition in (36) implies ω is a regulated Brownian motion in each market n_j. When $\omega(n_j, t) \in (\underline{\omega}_n(\hat{\omega}), \bar{\omega}_n(\hat{\omega}))$, only productivity shocks change ω, so

$$d\omega(n_j, t) = \frac{\theta_n - 1}{\theta_n} d \ln x(n_j, t) + \frac{q_n}{\theta_n} dt = \mu_n dt + \sigma_n dz(n_j, t), \tag{39}$$

where

$$\mu_n \equiv \frac{\theta_n - 1}{\theta_n} \mu_{n,x} + \frac{q_n}{\theta_n} \quad \text{and} \quad \sigma_n \equiv \frac{\theta_n - 1}{\theta_n} \sigma_{n,x},$$

i.e., $\omega(n_j, t)$ has drift μ_n and instantaneous standard deviation σ_n. When the thresholds $\underline{\omega}_n(\hat{\omega})$ and $\bar{\omega}_n(\hat{\omega})$ are finite, they act as reflecting barriers, since productivity shocks that would move ω outside the boundaries are offset by the entry and exit of workers.

4.3 The Value of a Worker

Consider a worker in a labor market n with log minimum wage $\hat{\omega}$. We can analyze the behavior of such a worker in isolation from the rest of the economy. For notational convenience, we suppress the dependence of the value function on sector-specific variables whenever there is no loss of clarity. Let $\{\eta_i\}_{i=1,2}$ be the roots of

$$\frac{1}{2}\sigma^2\eta^2 + \mu\eta - (\rho + \lambda) = 0, \tag{40}$$

where the values of μ and σ for this labor market's ω process are given by (39). Note that (28) and our assumption that $\theta > 1$ imply that $\frac{1}{2}\sigma^2 + \mu - (\rho + \lambda) < 0$, in other words $\eta_1 < 0 < 1 < \eta_2$.

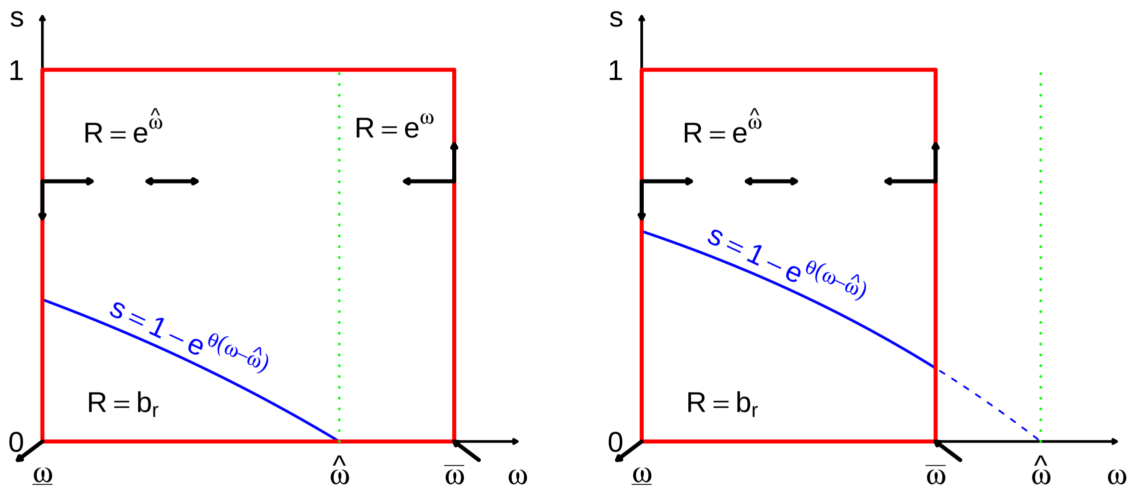
The worker's state is described by the log full employment wage in her labor market ω and her seniority s, as well as the characteristics of her labor market, including the log minimum wage, the stochastic process for productivity, and the substitutability of goods. But from the worker's perspective, it suffices to know that the log full employment wage is a regulated Brownian motion with endogenous, labor-market specific barriers $\underline{\omega} < \bar{\omega}$. Her seniority is her percentile in the tenure distribution in her labor market. When a worker arrives, she starts at $s = 0$. Subsequently when workers enter or exit the labor market, the seniority of all workers evolves so as to maintain a uniform distribution of s on $[0, 1]$. Thus s increases only when $\omega = \bar{\omega}$ and falls only when $\omega = \underline{\omega}$; Figure 1 shows the dynamics of ω and s. Each worker exits at the first time $\tau(\underline{\omega}, 0)$ that her state $(\omega(t), s(t))$ hits $(\underline{\omega}, 0)$, i.e. the first time she is the least senior worker in a market with log full employment wage $\underline{\omega}$. She also exits exogenously at rate $\lambda = q + \delta$, the sum of the quit rate and the rate at which the labor market shuts down.

To compute the value v of a worker in state (ω, s), let

$$R(\omega, s) = \begin{cases} e^{\omega} & \text{if } \omega \geq \hat{\omega} \\ e^{\hat{\omega}} & \text{if } \omega < \hat{\omega} \text{ and } s \geq 1 - e^{\theta(\omega - \hat{\omega})} \\ b_r & \text{if } \omega < \hat{\omega} \text{ and } s < 1 - e^{\theta(\omega - \hat{\omega})} \end{cases} \tag{41}$$

denote the flow payoff of a worker in each state, where we suppress the dependence of the elasticity of substitution θ, and hence the return function R, on labor market n. The

Figure 1: Dynamics of state vector (ω, s)



Left figure illustrates a case where the minimum wage is set at a moderate level, whereas the right figure illustrates the case where the minimum wage is set at a high level, so that it is binding all the time.

left-hand side of Figure 1 shows the flow payoff in (ω, s) space whenever the minimum wage $\hat{\omega}$ is moderate. If $\omega \geq \hat{\omega}$, all workers are employed at log wage ω. Otherwise, the most senior workers are employed at $\hat{\omega}$ and the less senior workers are unemployed and get leisure b_r. By construction $b_r \leq e^{\hat{\omega}}$, so employed workers are always weakly better off than unemployed workers. Workers in a particular labor market are indifferent between employment and unemployment only if $b_r = e^{\hat{\omega}}$ and $\omega \leq \hat{\omega}$.

The right-hand side of Figure 1 illustrates the state space when the minimum wage $\hat{\omega}$ is high, such that it is always binding. In such case, rest unemployment always exists in such a labor market, and it is equal to $1 - e^{\omega - \hat{\omega}}$.

Using the flow payoff R introduced in (41), we show in Appendix A.2.1 that the value of a worker in state (ω_0, s_0) in a market characterized by log minimum wage $\hat{\omega}$ and thresholds $\underline{\omega} < \bar{\omega}$ is

$$v(\omega_0, s_0; \hat{\omega}, \underline{\omega}, \bar{\omega}) = \mathbb{E}_{\omega_0, s_0}\left[\int_0^{\tau(\underline{\omega},0)} e^{-(\rho+\lambda)t}\big(R(\omega(t), s(t)) + \lambda \underline{v}\big)dt + e^{-(\rho+\lambda)\tau(\underline{\omega},0)}\underline{v}\right]. \tag{42}$$

Both the stopping time and the path of the state depend on the thresholds $\underline{\omega}$ and $\bar{\omega}$, while the period return function depends on $\hat{\omega}$. In equilibrium, workers must be willing to exit the labor market in state $(\underline{\omega}, 0)$ and to enter labor markets in state $(\bar{\omega}, 0)$. That is, $\underline{\omega}$ and $\bar{\omega}$ must satisfy

$$v(\underline{\omega}, 0; \hat{\omega}, \underline{\omega}, \bar{\omega}) = \underline{v} \tag{43}$$

$$v(\bar{\omega}, 0; \hat{\omega}, \underline{\omega}, \bar{\omega}) = \bar{v}, \tag{44}$$

where $\underline{v}$ and $\bar{v}$ are common to all labor markets and are determined by the leisure from search and inactivity and by the extent of search frictions; see equations (33)–(34). Workers must be willing to stay in labor markets in all other states:

$$v(\omega, s; \hat{\omega}, \underline{\omega}, \bar{\omega}) \geq \underline{v} \text{ for all } (\omega, s) \in [\underline{\omega}, \bar{\omega}] \times [0, 1]. \tag{45}$$

Note that in the presence of a binding minimum wage, workers in some states (ω, s) may attain a value strictly larger than $\bar{v}$. Workers from outside the labor market cannot move directly into such states because they do not have the requisite seniority.

In equilibrium, workers are just indifferent about exiting the labor market at the stopping time $\tau(\underline{\omega}, 0)$. This means that the value of a worker who stays in the labor market until she is hit by the exogenous quit shock is the same as the value of a worker who stays until either she is hit by the quit shock or the first time she reaches state $(\underline{\omega}, 0)$,

$$v(\omega_0, s_0; \hat{\omega}, \underline{\omega}, \bar{\omega}) = \mathbb{E}_{\omega_0, s_0}\left[\int_0^\infty e^{-(\rho+\lambda)t}\big(R(\omega(t), s(t)) + \lambda \underline{v}\big)dt\right], \tag{46}$$

when $(\underline{\omega}, \bar{\omega})$ solve equations (43) and (44) and all other workers follow the prescribed policy, exiting the first time they hit state $(\underline{\omega}, 0)$. The equivalence between the value functions in equations (42) and (46) simplifies our exposition.

4.4 Characterization of the Value Function

Appendix A.2.2 shows that v is twice differentiable on the interior of the state space, except at points where $R(\omega, s)$ is discontinuous, i.e. on the locus $s = 1 - e^{\theta(\omega - \hat{\omega})}$, where it is once differentiable. Appendix A.2.3 proves that v satisfies the following partial differential equations, for all (ω, s):

$$\rho v(\omega, s) = R(\omega, s) + \lambda(\underline{v} - v(\omega, s)) + \mu v_\omega(\omega, s) + \tfrac{1}{2}\sigma^2 v_{\omega\omega}(\omega, s). \tag{47}$$

At the highest and lowest wages and for all s,

$$v_\omega(\underline{\omega}, s) = v_s(\underline{\omega}, s)(1-s)\theta \tag{48}$$

$$v_\omega(\bar{\omega}, s) = v_s(\bar{\omega}, s)(1-s)\theta. \tag{49}$$

For a worker who is at the exit threshold,

$$v_\omega(\underline{\omega}, 0) = 0. \tag{50}$$

Finally, the highest level of seniority is an absorbing state until the worker exits the labor market, which ensures that

$$v_\omega(\underline{\omega}, 1) = v_\omega(\bar{\omega}, 1) = 0. \tag{51}$$

These act as boundary conditions and are used in our proof that the thresholds uniquely determine the value function. We summarize these results in the following proposition.

Proposition 3. For any $\bar{\omega} > \underline{\omega}$, $v(\omega, s)$ is uniquely determined by equations (47)–(51) and the condition that it is almost everywhere twice differentiable. $v(\omega, s)$ is strictly increasing in ω and strictly increasing in s if $\hat{\omega} > \underline{\omega}$ and independent of s otherwise.

Appendix A.2.4 gives closed form solutions for the value function in a typical labor market with an arbitrary minimum wage $\hat{\omega}$ and thresholds $\underline{\omega} \leq \bar{\omega}$. We display expressions for three cases depending on whether the minimum wage $\hat{\omega}$ is (i) smaller than $\underline{\omega}$, (ii) in the interval $(\underline{\omega}, \bar{\omega})$, or (iii) greater than or equal to $\bar{\omega}$. Monotonicity of the value function is proven in Appendix A.2.5 and Appendix A.2.6. Seniority matters only if the minimum wage sometimes binds, in the sense that unemployed workers are worse off than employed workers within the same market.

Interestingly, in the high minimum wage case case ($\hat{\omega} \geq \bar{\omega}$), our model — from the point of view of a worker — is isomorphic to the frictionless model we developed in section 2, as discussed below.

COROLLARY 2. When the minimum wage is sufficiently high (i.e. when $\hat{\omega} \geq \bar{\omega}$), the worker's value function only depends on $y_t \equiv e^{\theta(\omega_t - \hat{\omega})}(1-s_t)^{-1}$, i.e. her seniority relative to a rescaled version of the log-full employment wage ω_t. When $y \geq 1$, the worker's seniority is high enough and the worker is employed at the minimum wage. When $y < 1$, the worker rests, and exits the labor market when $y < e^{\theta(\underline{\omega} - \hat{\omega})}$. The dynamics of y_t are identical to the dynamics of $(1-s_t)^{-1}$ in the frictionless model of Section 2.

This corollary can be interpreted as follows. When the minimum wage always binds, the flow payoff to a worker in a labor market is either $e^{\hat{\omega}}$ (when the worker is employed), or b_r (when the worker is rest unemployed) — i.e. identical to the flow payoff in the frictionless model. Positive productivity shocks increase wage pressure, and eventually lead to entry of new workers once ω_t hits $\bar{\omega}$, which moves a worker's seniority higher. Similarly, with negative productivity shocks, the wage pressure diminishes, and after a sufficient number of negative shocks, ω_t reaches $\underline{\omega}$ and workers start leaving. Thus, while the worker's valuation is isomorphic to its valuation in the frictionless model, the two models are not observationally equivalent. In the frictionless model, productivity shocks are always associated with inflows in and outflows out of such labor market, while in the model with search friction, the labor market does not see any entry or exits when the log full employment wage $\omega(t)$ is inside $(\underline{\omega}, \bar{\omega})$.

Proposition 3 establishes the existence of an equilibrium for any thresholds and minimum wage $\underline{\omega} \leq \hat{\omega} \leq \bar{\omega}$ if the values of the parameters (b_i, b_s, α) are such that the values of $\underline{v}$ and $\bar{v}$ satisfy (43) and (44). Since the observable implication for the labor market depends only on $(\underline{\omega}, \hat{\omega}, \bar{\omega})$, we can use this result to find the implied values for (b_i, b_s, α) to rationalize such equilibrium. In the next section we turn to the inverse mapping: fixing the parameters that determine $\underline{v}, \bar{v}$ and fixing $\hat{\omega}$ we show that there exist thresholds $\underline{\omega}, \bar{\omega}$ for which there is an equilibrium.

4.5 Existence of Equilibrium

In this section we establish the existence of an equilibrium given model parameters that determine the search and inaction values $\underline{v}, \bar{v}$.

Proposition 4. Fix the value of inaction and search satisfying $0 < \underline{v} < \bar{v}$. Remember that $\hat{\omega} \equiv \max\left(\ln \frac{\hat{w}}{C}, \ln b_r\right)$. There are two thresholds $({}_*\hat{\omega}, \hat{\omega}^*)$, with ${}_*\hat{\omega} < \ln b_i < \hat{\omega}^*$, such that:

- If $\hat{\omega} <_* \hat{\omega}$, there exists unique equilibrium thresholds $(\underline{\omega}^*, \bar{\omega}^*)$, with $\hat{\omega} < \underline{\omega}^* < \bar{\omega}^*$, and which satisfy equations (43) and (44). In such labor market, there is no rest unemployment;
- If $\hat{\omega} > \hat{\omega}^*$, there exists unique equilibrium thresholds $(\underline{\omega}(\hat{\omega}), \bar{\omega}(\hat{\omega}))$, with $\underline{\omega}(\hat{\omega}) < \bar{\omega}(\hat{\omega}) < \hat{\omega}$, and which satisfy equations (43) and (44):

$$\bar{\omega}(\hat{\omega}) = \hat{\omega} - \frac{1}{\eta_2} \ln\left(\frac{e^{\hat{\omega}} - b_r}{e^{\hat{\omega}^*} - b_r}\right) \tag{52}$$

$$\underline{\omega}(\hat{\omega}) = \hat{\omega} - \frac{1}{\eta_2} \ln\left(\frac{e^{\hat{\omega}} - b_r}{b_i - b_r}\right) \tag{53}$$

  In such labor market, either workers are employed at the minimum wage $\hat{w}$, or they are rest unemployed;
- If $\hat{\omega} \in ({}_*\hat{\omega}, \hat{\omega}^*)$, there exists equilibrium thresholds $(\underline{\omega}(\hat{\omega}), \bar{\omega}(\hat{\omega}))$, with $\underline{\omega}(\hat{\omega}) < \hat{\omega} < \bar{\omega}(\hat{\omega})$, and which satisfy equations (43) and (44). In such labor market, when $\omega > \hat{\omega}$, all workers are employed, while when $\omega < \hat{\omega}$, a fraction $1 - e^{\theta(\omega - \hat{\omega})}$ of workers are rest unemployed, while the remaining workers are employed and earn the minimum wage.

Proposition 4 tells us that the mapping is invertible: given $\underline{v}$ and $\bar{v}$, we can find $\underline{\omega}$ and $\bar{\omega}$. The case $\hat{\omega} <_* \hat{\omega}$ is the situation analyzed in Alvarez and Shimer (2011). Their Proposition 2 shows that, everything else fixed, there exists a threshold $\bar{b}_r > 0$ such that if $b_r < \bar{b}_r$, there is no rest unemployment. In that case, there exists a unique equilibrium characterized by thresholds $\bar{\omega}^* > \underline{\omega}^* > \ln b_r$ where workers enter and exit labor markets so as to regulate wages in $[\underline{\omega}^*, \bar{\omega}^*]$. These thresholds and the associated value function satisfy (43)–(46). The cutoff ${}_*\hat{\omega}$ is then simply equal to the reflecting boundary $\underline{\omega}^*$ in the absence of minimum wage[4].

When the minimum wage is sufficiently high (i.e. when $\hat{\omega} > \hat{\omega}^*$), there is always rest unemployment in such labor market, and the minimum wage always binds. In such case, we prove in Appendix A.2.7 that the equilibrium is unique. The cutoff $\hat{\omega}^*$ is the solution to a non-linear equation that depends only on the model parameters $b_i, b_r, b_s, \alpha, \rho, \lambda, \sigma, \mu$. The distance between the two reflecting barriers $\bar{\omega}(\hat{\omega}) - \underline{\omega}(\hat{\omega})$ no longer depends on $\hat{\omega}$, and is instead equal to $\frac{1}{\eta_2} \ln\left(\frac{e^{\hat{\omega}^*} - b_r}{b_i - b_r}\right)$ (i.e. it depends only on the deep model parameters).

For intermediate values of the minimum wage (i.e. when ${}_*\hat{\omega} > \hat{\omega} > \hat{\omega}^*$), we conjecture that the thresholds are unique but do not have a proof.

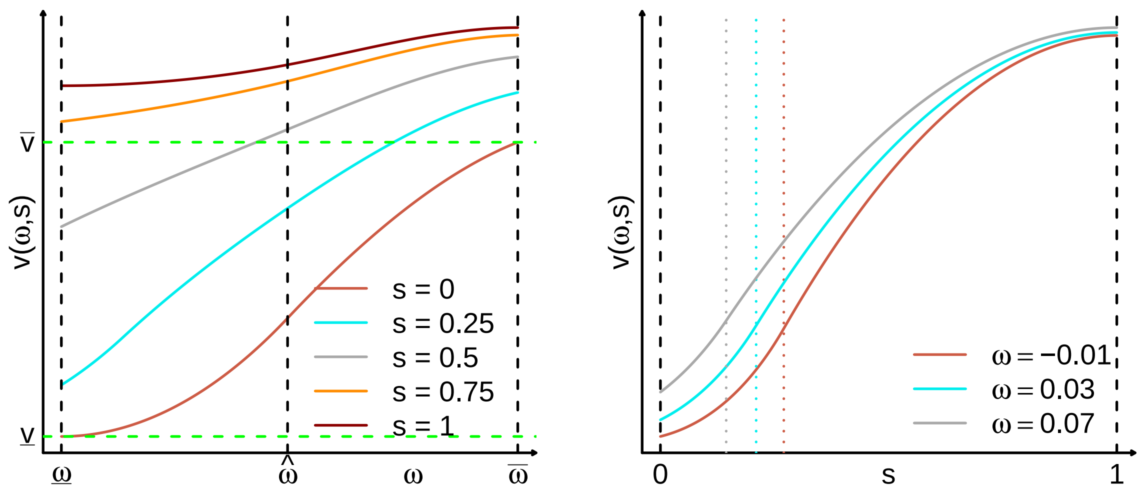
4.6 Illustration

In this section, we provide a graphical illustration of all the calculations performed until now for intermediate values of the minimum wage — i.e. for the case where the minimum wage binds sometimes, but not all the time. Figure 2 shows the value function $v(\omega, s)$ for parameters in line with those of Alvarez and Shimer (2011), when $\hat{\omega} = 0.15$, first as a function of ω for different seniorities (left-hand side), and then as a function of seniority s for different values of ω (right-hand side).[5] More senior workers are always better off than less senior workers and all workers are better off when the log full employment wage ω is higher, although more senior workers' value function is less sensitive to ω. After a sequence of bad productivity shocks pushing ω towards $\underline{\omega}$, the most junior worker entering has a value $v(\underline{\omega}, 0) = \underline{v}$, making her indifferent between staying in such labor market or exiting. Similarly, after a sequence of good productivity shocks pushing ω towards $\bar{\omega}$, the most junior worker entering has a value $v(\bar{\omega}, 0) = \bar{v}$.

[4] One might have imagined that a binding minimum wage simply raised the lower threshold for ω so $\underline{\omega} = \hat{\omega}$. This is not the case. Since the standard deviation of productivity per unit of time explodes when the time horizon is short, the option value of entering rest unemployment, at least briefly, always exceeds the option value of immediately exiting the labor market when productivity falls too far.

[5] We consider a labor market with an elasticity of substitution $\theta = 2$, households with discount rate $\rho = 0.05$, leisure value of inactivity $b_i = 1$, search costs $\kappa = 2$, job finding rate for searchers to $\alpha = 3.2$, so that $\underline{v} = 20$ and $\bar{v} = 22$. We set leisure from rest unemployment to $b_r = 0.7$, the standard deviation of wages at $\sigma = 0.12$ and the quit rate at 0.04, so that $\mu = q/\theta - \theta\sigma^2/2 \approx 0.0056$ and we can focus on the limit as $\delta \to 0$. In the absence of a minimum wage, we find that $\underline{\omega} = -0.258$, higher than $\ln b_r = -0.357$. Therefore any minimum wage below this level has no effect.

Figure 2: Value function $v(\omega, s)$

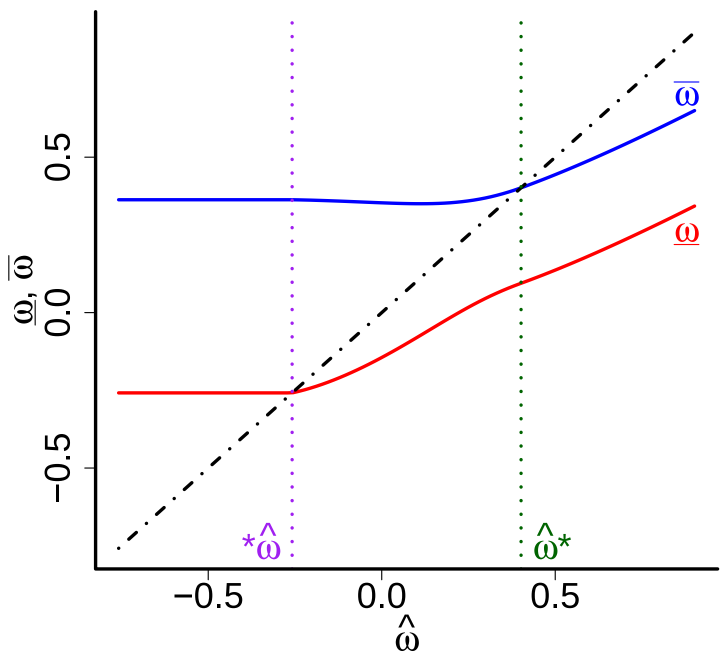


Parameters are $\theta = 2$, $\rho = 0.05$, $b_i = 1$, $b_r = 0.7$, $\alpha = 3.2$, $\kappa = 2$, $q = 0.04$, $\mu = 0.0056$, $\sigma = 0.12$ and minimum wage $\hat{\omega} = 0.15$. Dash vertical black lines on the left-hand side indicate the reflecting boundaries $\underline{\omega}$ and $\bar{\omega}$, whereas they indicate the boundaries of the state space $s = 0$ and $s = 1$ on the right-hand side. Dash horizontal green lines on the left-hand side show the worker value at exit $\underline{v}$ and at entry $\bar{v}$. Dotted colored lines on the right-hand side indicate critical seniority values $s = 1 - e^{\theta(\omega - \hat{\omega})}$ below which the worker is rest-unemployed.

The right-hand side of the figure shows the value function $v(\omega, \cdot)$ for specific values of ω, with the threshold $s = 1 - e^{\theta(\omega - \hat{\omega})}$ at which the worker switches from being rest-unemployed to being employed at the minimum wage whenever $\omega < \hat{\omega}$ (dotted lines).

Figure 3 shows how the thresholds change as functions of the minimum wage. The left-most dotted purple line is ${}_*\hat{\omega}$, below which there is no equilibrium with the minimum wage ever binding. The right-most dotted dark green line is $\hat{\omega}^*$, above which the only equilibrium exhibits always-binding minimum wage, and where there is always rest-unemployment. The lower bound $\underline{\omega}$ increases in $\hat{\omega}$, with a slope less than 1. Put differently, when the minimum wage is higher, the maximum number of workers willing to stay in the labor market is smaller for any value of productivity. On the other hand, the upper bound initially falls with $\hat{\omega}$, indicating that a modest degree of monopolization attracts workers to the labor market for a given level of productivity. This is true even though the last entrant to the union is the first worker laid off.

Figure 3: Thresholds



Parameters are $\theta = 2$, $\rho = 0.05$, $b_i = 1$, $b_r = 0.7$, $\alpha = 3.2$, $\kappa = 2$, $q = 0.04$, $\mu = 0.0056$, and $\sigma = 0.12$. Dotted vertical purple line indicates the threshold $_*\hat{\omega}$ below which the minimum wage never binds; dotted vertical dark green lines shows the threshold $\hat{\omega}^*$ above which the minimum wage binds all the time. Dot-dash black line indicates the 45 degree line.

4.7 Aggregation

Consider a sector n and minimum wage rate $\hat{\omega}$. Denote the related thresholds by $\underline{\omega}_n(\hat{\omega})$ and $\bar{\omega}_n(\hat{\omega})$. Given these, we can compute the fraction of workers at each value of $\omega \in [\underline{\omega}_n(\hat{\omega}), \bar{\omega}_n(\hat{\omega})]$. Note that this is different from the fraction of labor markets at each value of ω, since there are typically more workers in labor markets with a higher log full employment wage.

PROPOSITION 5. The steady state density of workers' log full employment wage in sector

n, minimum wage $\hat{\omega}$ is

$$f_n(\omega;\hat{\omega}) = \frac{\sum_{i=1}^{2} |\xi_{i,n}+\theta_n| e^{\xi_{i,n}(\omega-\underline{\omega}_n(\hat{\omega}))}}{\sum_{i=1}^{2} |\xi_{i,n}+\theta_n| \frac{e^{\xi_{i,n}(\bar{\omega}_n(\hat{\omega})-\underline{\omega}_n(\hat{\omega}))}-1}{\xi_i}}, \tag{54}$$

where $\xi_{1,n} < \xi_{2,n}$ solve the characteristic equation $\delta_n + q_n = -\mu_n \xi_n + \frac{\sigma_n^2}{2}\xi_n^2$ and $\underline{\omega}_n(\hat{\omega}) < \bar{\omega}_n(\hat{\omega})$ are the thresholds for that sector and minimum wage.

The proof of this result is identical to Proposition 3 in Alvarez and Shimer (2011) and hence omitted. That proposition also shows how to close the model via the computation of the number of workers across labor markets and of the consumption of each good, results that we do not repeat here. Note that under condition (28), $\xi_{1,n} \leq -\theta_n$ and $\xi_{2,n} > 0$.

Using this result, we can compute the rest and search unemployment rates for each sector n and minimum wage $\hat{w}$. To reduce the notation, we suppress the dependence of the thresholds on n and $\hat{w}$. If $\hat{\omega} \leq \underline{\omega}$, there is no rest unemployment in any such sector. Otherwise when $\omega < \hat{\omega}$, all workers with seniority $s < 1 - e^{\theta_n(\omega-\hat{\omega})}$ are rest unemployed. This gives the rest unemployment rate in such sector. Integrating across labor markets using (54) gives the sector- and minimum wage-specific rest unemployment rate

$$\frac{U_{r,n}(\hat{w})}{L_n(\hat{w})} = \int_{\underline{\omega}}^{\min\{\hat{\omega},\bar{\omega}\}} \left(1 - e^{\theta_n(\omega-\hat{\omega})}\right) f_n(\omega;\hat{\omega})\, d\omega.$$

where $U_{r,n}(\hat{\omega})$ is the number of rest unemployed and $L_n(\hat{\omega})$ is the number of (employed or unemployed) workers in such sector. This gives

$$\frac{U_{r,n}(\hat{w})}{L_n(\hat{w})} = \frac{\sum_{i=1}^{2} |\xi_{i,n}+\theta_n| \left[\frac{e^{\xi_{i,n}(\bar{\omega}\wedge\hat{\omega}-\underline{\omega})}-1}{\xi_{i,n}} - e^{-\theta(\hat{\omega}-\underline{\omega})}\frac{e^{(\xi_{i,n}+\theta_n)(\bar{\omega}\wedge\hat{\omega}-\underline{\omega})}-1}{\xi_{i,n}+\theta_n}\right]}{\sum_{i=1}^{2} |\xi_{i,n}+\theta| \frac{e^{\xi_{i,n}(\bar{\omega}-\underline{\omega})}-1}{\xi_{i,n}}}. \tag{55}$$

Using this equation, we can easily compute how the level of the minimum wage affects the unemployment rate within a sector and how a given minimum wage affects the unemployment rate in different sectors.

Consider the search unemployed connected to a particular sector n and minimum wage $\hat{w}$. Let $N_{s,n}(\hat{\omega})$ be the number of workers that leave their labor market per unit of time, either because conditions are sufficiently bad or because they exogenously quit or because their labor market has exogenously shut down. As in Alvarez and Shimer (2011), this satisfies

$$N_{s,n}(\hat{\omega}) = \left(\frac{\theta_n \sigma_n^2}{2} f_n(\underline{\omega}; \hat{\omega}) + \delta_n + q_n \right) L_n(\hat{\omega}). \tag{56}$$

The first term gives the fraction of workers who leave their labor market to keep ω above $\underline{\omega}$, while the second term is related to exogenous departures. In steady state, the fraction of workers leaving their labor markets must balance the fraction of workers arriving in such labor markets. The latter is given by the fraction of workers engaged in searching for this sector and minimum wage, $U_{s,n}(\hat{\omega})$, times the rate at which they arrive to a labor market α, so $\alpha U_{s,n}(\hat{\omega}) = N_{s,n}(\hat{\omega})$. Using (56) and (54) delivers the ratio of search unemployment to workers in a sector:

$$\frac{U_{s,n}(\hat{\omega})}{L_n(\hat{\omega})} = \frac{1}{\alpha} \left(\frac{\theta_n \sigma_n^2}{2} \frac{\xi_{2,n} - \xi_{1,n}}{\sum_{i=1}^{2} |\theta_n + \xi_{i,n}| \frac{e^{\xi_{i,n}(\bar{\omega} - \underline{\omega})} - 1}{\xi_{i,n}}} + \delta_n + q_n \right) \tag{57}$$

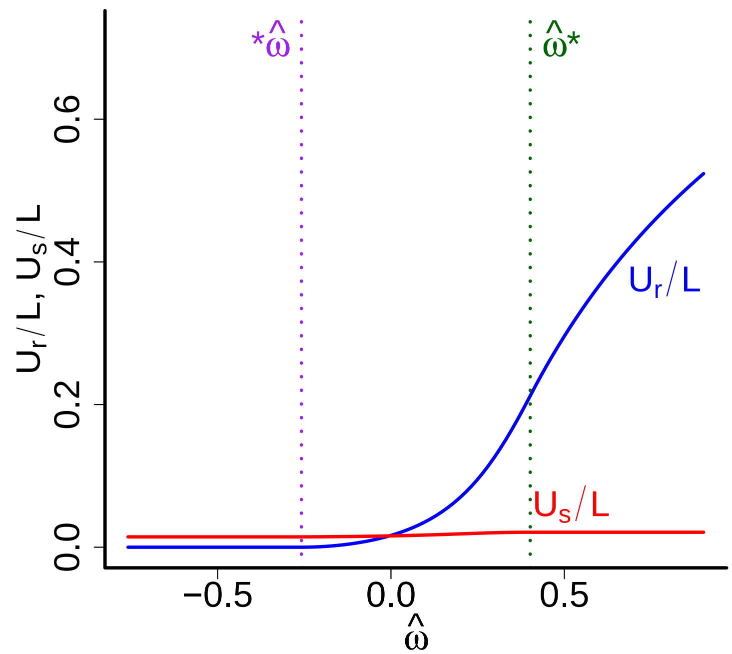
To compute the aggregate rest and search unemployment rates, simply aggregate across minimum wages and sectors.

Using our example from Section 4.6, Figure 4 illustrates how rest and search unemployment rates in a sector vary with $\hat{\omega}$. Initially there is no rest unemployment, although search unemployment is necessary to sustain the sector. As the minimum wage rises, the rest unemployment rate starts to increase while the search unemployment rate is approximately unchanged; union-mandated minimum wages thus provide a powerful mechanism for generating rest unemployment.

4.8 Hazard Rate of Exiting Unemployment

When there is no rest unemployment, the hazard of exiting unemployment is simply α. This section characterizes the hazard of exiting unemployment when there is rest unemployment, $\hat{\omega} > \underline{\omega}$, but not in the best markets, $\bar{\omega} > \hat{\omega}$. We will show that this haz-

Figure 4: Unemployment as a Function of Minimum Wage



Parameters are $\theta = 2$, $\rho = 0.05$, $b_i = 1$, $b_r = 0.7$, $\alpha = 3.2$, $\kappa = 2$, $q = 0.04$, $\mu = 0.0056$, and $\sigma = 0.12$. Dotted vertical purple line indicates the threshold ${}_*\hat{\omega}$ below which the minimum wage never binds; dotted vertical dark green lines shows the threshold $\hat{\omega}^*$ above which the minimum wage binds all the time.

ard is downward sloping and exhibits duration dependence; the low hazard of exiting long-term unemployment may be important for understanding the coexistence of many workers who move easily between jobs and a relatively small number of workers who suffer extended unemployment spells (see Juhn, Murphy and Topel (1991)).

We determine the hazard of ending an unemployment spell of duration t, denoted $h(t)$, in two steps. First, we study the extent to which this hazard depends on the seniority of the worker at the time she enters unemployment.

Lemma 1. *$h(t)$ is the same for all workers in a labor market, regardless of seniority.*

The proof of Lemma 1, in Appendix A.2.8.1, shows that this hazard depends only the distance between $\underline{\omega}$ and $\hat{\omega}$, but not on the seniority s or full-employment wage ω at the time such worker enters unemployment.

A worker entering unemployment either becomes (i) search unemployed (upon a quit shock or a labor market closure), or (ii) rest unemployed (if conditions in her labor market have sufficiently deteriorated). In the first case, she regains employment after searching, at rate α.[6] In the second case, she regains employment either when (a) local labor market conditions improve enough for her to reenter employment — occurring at a duration-dependent hazard $\hat{h}_r(t)$ — or (b) local labor market conditions have deteriorated enough that she ends up endogenously leaving such market — occurring at a duration-dependent hazard $\underline{h}_r(t)$ — and finds a more attractive labor market after searching. This discussion is formalized in our next proposition.

PROPOSITION 6. *Let $u_r(t)$ (resp. $u_s(t)$) be the duration-contingent rest (resp. search) unemployment probability; $h(t)$ is equal to*

$$h(t) = \hat{h}_r(t)\frac{u_r(t)}{u_r(t)+u_s(t)} + \alpha\frac{u_s(t)}{u_r(t)+u_s(t)},$$

where the duration-contingent unemployment probabilities solve a system of two ordinary differential equations with time-varying coefficients (see equations (A40) in Appendix A.2.8.2). The sub-hazard rates $\underline{h}_r(t), \hat{h}_r(t)$ admit series expansions with closed form expressions (see equations (A42)-(A43) in Appendix A.2.8.3). Asymptotically,

$$\lim_{t\to 0} th(t) = \frac{1}{2} \tag{58}$$

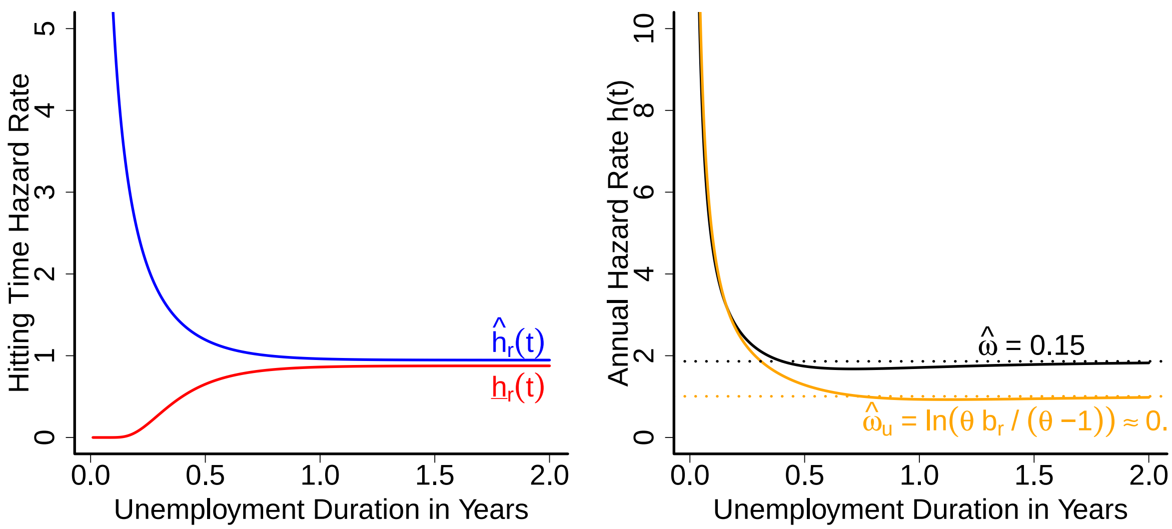
$$\lim_{t\to+\infty} h(t) = \min\left(\alpha, \delta + q + \frac{1}{2}\left(\frac{\mu^2}{\sigma^2} + \frac{\pi^2\sigma^2}{(\hat{\omega}-\underline{\omega})^2}\right)\right). \tag{59}$$

The differential equations driving the duration-dependence of the unemployment rates $u_r(t)$ and $u_s(t)$ encode the fact that (i) rest-unemployed individuals transition out

[6] Here we use the assumption that $\hat{\omega} < \bar{\omega}$ so when a search-unemployed worker finds a market, she goes to work immediately. Implicitly, we are also assuming that as soon as a worker leaves a labor market, she immediately goes into search, rather than becoming inactive. This assumption can be microfounded for instance if each member k of the representative household has a member-specific flow utility of inactivity b_{ik}. In that case, the members with highest value b_{ik} will end up always inactive, while the members with the lowest value b_{ik} will always be immediately deployed into search, once they exit a given labor market. We can regard our model, as the limit case when the dispersion of b_{ik} across members vanishes.

of rest unemployment at rate $\delta + q + \underline{h}_r(t) + \hat{h}_r(t)$, and (ii) search-unemployed individuals find jobs at rate α while rest-unemployed individuals transition into search unemployment at rate $\delta + q + \underline{h}_r(t)$. The series expansions obtained for the sub-hazard rates $\underline{h}_r(t)$ and $\hat{h}_r(t)$ rely on standard results for hitting times of a regulated Brownian motion.

Figure 5: Hazard Rates



Left figure illustrates the hitting time hazard rates $\hat{h}_r(t)$ (blue line) and $\underline{h}_r(t)$ (red line), for a minimum wage $\hat{\omega} = 0.15$. Right figure shows the employment hazard rate $h(t)$ for a minimum wage $\hat{\omega} = 0.15$ (black line) as well as for the minimum wage $\hat{\omega}_u = 0.34$ (orange line) that would be set optimally by a union (see Section 5). In both figures, parameters are $\theta = 2$, $\rho = 0.05$, $b_i = 1$, $b_r = 0.7$, $\alpha = 3.2$, $\kappa = 2$, $q = 0.04$, $\mu = 0.0056$ and $\sigma = 0.12$.

Proposition 6 suggests that the short term hazard of exiting unemployment behaves as $1/(2t)$. Intuitively, consider a worker on the threshold of rest unemployment, $s = 1 - e^{\theta(\omega - \hat{\omega})}$. After a short time interval—short enough that the variance of the Brownian motion dominates the drift—there is a $\frac{1}{2}$ probability that ω has increased, so the worker is reemployed, and a $\frac{1}{2}$ chance it has fallen. But a $\frac{1}{2}$ probability over any horizon t implies a hazard rate $1/2t$. Thus our model predicts that unionized workers experience many short spells of unemployment, which perhaps can be interpreted as temporary layoffs.

The asymptotic hazard in (59) is decreasing in the distance $\hat{\omega} - \underline{\omega}$. Thus, the hazard rate out of unemployment might be low at long unemployment durations if search frictions are high enough; in other words, unionized workers will sometimes remain unemployed for years with little chance of reemployment.

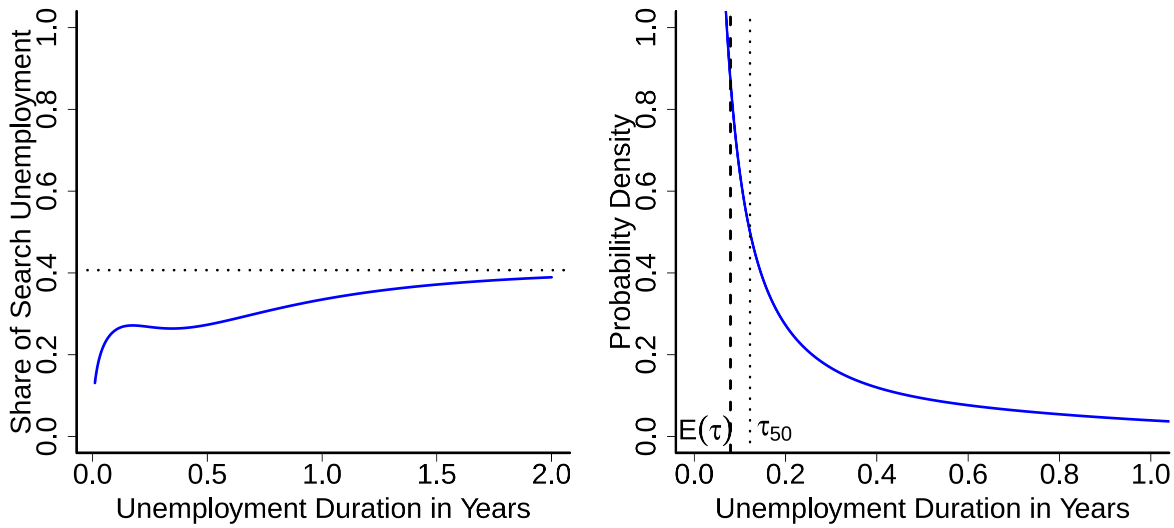
The left-hand side of Figure 5 illustrates $\underline{h}_r(t)$ and $\hat{h}_r(t)$ for parameters considered in Section 4.6. Conditional on just having lost her job, a rest-unemployed worker has an arbitrarily high hazard rate of regaining employment in the next instant, while it has zero probability of leaving (endogenously) its labor market. Conditional on having been unemployed for some time, both hitting time hazard rates stabilize at long run values.

The right-hand side of Figure 5 shows the annual hazard rate of finding a job in our baseline calibration (in solid black, with its asymptote in dotted black), as well as in case of a higher minimum wage $\hat{\omega}_u = 0.34$, which would be set optimally by a union (see Section 5). $h(t)$ behaves similarly to $\hat{h}_r(t)$ at short duration. With a low minimum wage, few workers get trapped in long-term unemployment because the gap between the minimum wage $\hat{\omega}$ and the exit threshold $\underline{\omega}$ is small. That is, most workers either quickly find a job or exit the labor market. With a higher (monopoly) union wage $\hat{\omega}_u > \hat{\omega}$, more workers get stuck in long-term unemployment. This tends to increase unemployment duration, and overall unemployment. In a labor market with such a high minimum wage, the efficiency of search affects the hazard of exiting long-term unemployment only indirectly, through its influence on the distance between the rest unemployment boundaries $\hat{\omega} - \underline{\omega}$.

Since the rest unemployed find jobs so quickly at the start of an unemployment spell, the share of searchers among the unemployed grows rapidly (left-hand side of Figure 6). After this point, however, the hazard of exiting rest unemployment falls below the hazard of exiting search unemployment and so the share of searchers stabilizes. The characterization of $h(t)$ allows us to compute the full distribution of unemployment duration τ, using $\Pr(\tau \leq t) = 1 - \exp\left(\int_0^t h(s)ds\right)$. We illustrate such distribution on the right-hand side of Figure 6.

Our finding of a constant hazard rate for workers in search unemployment and a decreasing hazard rate for workers in rest unemployment is consistent with empirical evidence. Katz and Meyer (1990) show that the empirical decline in the job finding hazard rate is concentrated among workers on temporary layoff. Moreover, they find that workers who expect to be recalled to a past employer and are not—in our model, workers who end a spell of rest unemployment by searching for a new labor market, at

Figure 6: Searchers Amongst Unemployed and Unemployment Duration



Left figure illustrates the fraction of searchers amongst unemployed workers. Right figure shows the full distribution of unemployment duration, as well as its mean and median. In both figures, parameters are $\theta = 2$, $\rho = 0.05$, $b_i = 1$, $b_r = 0.7$, $\alpha = 3.2$, $\kappa = 2$, $q = 0.04$, $\mu = 0.0056$, $\sigma = 0.12$ and minimum wage $\hat{\omega} = 0.15$.

hazard $\underline{h}_r(t) + \delta + q$—experience longer unemployment duration than observationally equivalent workers who immediately entered search unemployment — in our model, workers experiencing a δ or q shock. Starr-McCluer (1993) finds that the hazard of exiting unemployment is decreasing for workers who move to a job that is similar to their previous one (rest unemployed) while it is actually increasing for workers who move to a different type of job (search unemployed).

4.9 Random Allocation vs. Seniority Rule

In Section 2.6 we compared the equilibrium unemployment rate with seniority rule with that which would be prevalent in a world where jobs are allocated randomly to workers, and we showed that seniority rule lowers unemployment. We revisit this analysis in the presence of the search frictions of our full model.

Consider a labor market with a minimum wage, search frictions, but where jobs are allocated randomly — rather than subject to a seniority rule. We look for an equilibrium of this environment in which the full employment wage ω is regulated between two barriers $\underline{\omega}_r, \bar{\omega}_r$. Similar to our model with seniority rule, when $\omega(t)$ falls sufficiently following a sequence of bad productivity shocks, workers exit such market; instead, when $\omega(t)$ increases sufficiently after a sequence of good shocks, the labor market is so attractive that new workers enter. In such an equilibrium, the unemployment rate $u(\omega)$ in a market with full employment wage ω is either zero in case $\omega \geq \hat{\omega}$, or $1 - e^{\theta(\omega - \hat{\omega})}$ whenever the minimum wage is binding. When the minimum wage is binding and some workers are rest-unemployed, a worker is employed at the minimum wage with probability $1 - u\left(\omega(t)\right)$ and is rest-unemployed otherwise. With random allocation of workers into jobs, the flow payoff for a given worker is

$$R(\omega) = \begin{cases} e^{\omega} & \text{if } \omega \geq \hat{\omega} \\ \left(1 - u(\omega)\right) e^{\hat{\omega}} + u(\omega) b_r & \text{if } \omega < \hat{\omega} \end{cases} \tag{60}$$

The value of a worker in a market with full employment wage $\omega = \omega_0$ characterized by log minimum wage $\hat{\omega}$ and thresholds $\underline{\omega}_r < \bar{\omega}_r$ is

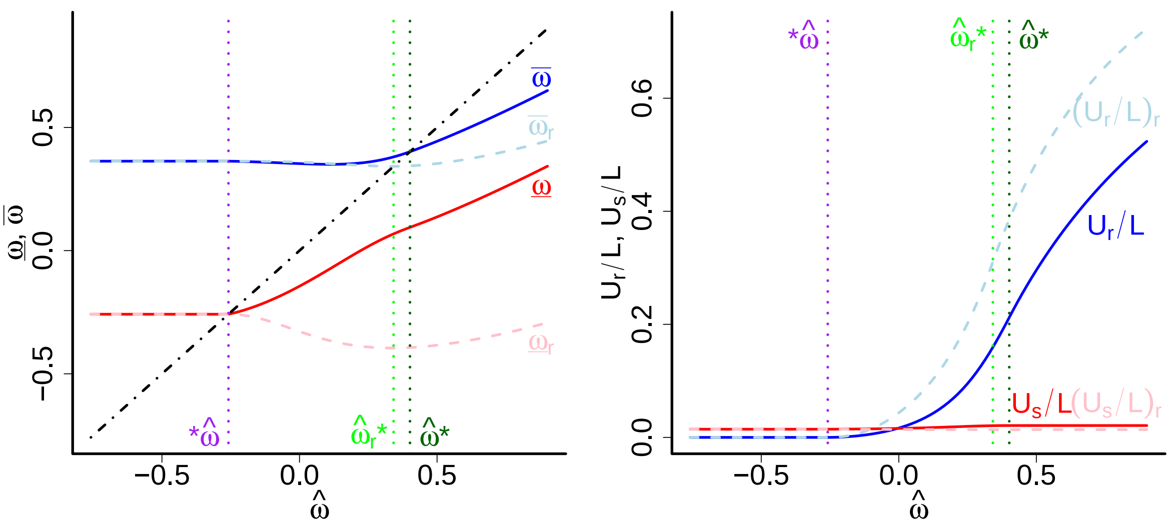
$$v(\omega_0; \hat{\omega}, \underline{\omega}_r, \bar{\omega}_r) = \mathbb{E}_{\omega_0}\left[\int_0^{\tau(\underline{\omega}_r)} e^{-(\rho+\lambda)t}\big(R(\omega(t)) + \lambda \underline{v}\big) dt + e^{-(\rho+\lambda)\tau(\underline{\omega}_r)} \underline{v}\right]$$

Thus, all workers in a given labor market have the same continuation value v, and their relative seniority becomes irrelevant.

Using a method similar to what was employed previously, Appendix A.2.9 provides analytic expressions for the value function in the case where the minimum wage is either (i) low enough that it never binds — in which case the equilibrium we are studying is identical to that studied in Alvarez and Shimer (2011), (ii) intermediate so that it binds sometimes, and (iii) high enough so that it binds at all times. The crucial distinction between our setup with random allocation rule and that with seniority rule is that reflecting boundaries $\underline{\omega}_r$ (at which point workers exit the market) and $\bar{\omega}_r$ (at which point workers enter the market) will differ from those prevalent in the seniority rule environment. Similar to Proposition 4 (omitted in the paper), one could show the existence of limiting minimum wage thresholds ${}_*\hat{\omega}_r$ and $\hat{\omega}_r^*$, with ${}_*\hat{\omega}_r = *\hat{\omega}$, which distinguish the three possible types of equilibria of this model:

- For low values of the minimum wage, $\hat{\omega} <_* \hat{\omega}$, the equilibrium with random allocation of jobs is identical to that with a seniority rule, the minimum wage never binds and there is no rest unemployment;
- For high values of the minimum wage, $\hat{\omega} > \hat{\omega}_r^*$ there exists equilibrium thresholds $(\underline{\omega}_r(\hat{\omega}), \bar{\omega}_r(\hat{\omega}))$, with $\underline{\omega}_r(\hat{\omega}) < \bar{\omega}_r(\hat{\omega}) < \hat{\omega}$, and in such labor market, the minimum wage binds at all times and there is always some rest unemployment, i.e. $u(\omega) > 0$ always;
- For intermediate values of the minimum wage, ${}_*\hat{\omega}_r < \hat{\omega} < \hat{\omega}_r^*$, there exists equilibrium thresholds $(\underline{\omega}_r(\hat{\omega}), \bar{\omega}_r(\hat{\omega}))$, with $\underline{\omega}_r(\hat{\omega}) < \hat{\omega} < \bar{\omega}_r(\hat{\omega})$, and in such labor market, the minimum wage binds whenever $\omega > \hat{\omega}$.

Figure 7: Optimal Boundaries and Unemployment Rates



Parameters are $\theta = 2$, $\rho = 0.05$, $b_i = 1$, $b_r = 0.7$, $\alpha = 3.2$, $\kappa = 2$, $q = 0.04$, $\mu = 0.0056$, and $\sigma = 0.12$ when we vary the minimum wage $\hat{\omega}$. Dotted vertical purple line indicates the threshold ${}_*\hat{\omega}$ below which the minimum wage never binds; dotted vertical dark (resp. light) green lines shows the threshold $\hat{\omega}^*$ (resp. $\hat{\omega}_r^*$) above which the minimum wage binds all the time in the seniority rule model (resp. the random allocation rule model).

The left-hand side of Figure 7 illustrates the behavior of the regulated boundaries $\underline{\omega}_r, \bar{\omega}_r$ in the model with random allocation vs. those in the seniory rule model, as a function of the minimum wage $\hat{\omega}$. For low minimum wages $\hat{\omega} <_* \hat{\omega}_r =_* \hat{\omega}$, these boundaries are identical. As the minimum wage increases, both $\underline{\omega}_r$ and $\bar{\omega}_r$ are lower than their counterpart in the seniority rule equilibrium. This suggests that — controlling for productivity — workers are attracted more easily into a labor market that has a random allocation rule, but that workers stick around longer in labor markets with bad productivities before exiting (relative to the case with seniority rule). Moreover, the threshold minimum wage $\hat{\omega}_r^*$ above which the minimum wage starts binding at all times is below its counterpart $\hat{\omega}^*$ in an environment with seniority rule. Lastly, the right-hand side of Figure 7 suggests that these lower reflecting boundaries lead to higher rest unemployment in the random allocation model, relative to the environment with a seniority rule; while search unemployment seems slightly lower with a random allocation rule, total unemployment is worse at all levels of the minimum wage $\hat{\omega}$. These calculations thus confirm the intuition built in our framework of Section 2 without search frictions: seniority rule results in labor markets with lower levels of unemployment.

5 Union Objective Function

In this final section, we consider a monopoly union representing the $\ell(n_j, t)$ workers in labor market n_j at time t. The union's objective is to maximize the total flow utility of those workers,

$$e(n_j, t)w(n_j, t)\frac{1}{\bar{C}} + \big(\ell(n_j, t) - e(n_j, t)\big)b_r,$$

where $e(n_j, t)$ is the measure of workers who are employed, $w(n_j, t)$ is the wage, and $1/\bar{C}$ is the marginal utility of consumption. For example, we can think of the union setting the wage and then letting competitive firms determine how many workers to hire. Our setup thus departs from Grossman (1983), who instead maximizes the utility of the median union member, mimicking the internal political process of many unions.

From the analysis in Section 3.3, we know that employment is

$$e(n_j,t) = \min\left\{\ell(n_j,t), \frac{\bar{C}(t)^{\theta_n}\left(Ax(n_j,t)\right)^{\theta_n-1}}{C(n,t)^{\theta_n-1}w^{\theta_n}}\right\}.$$

The solution to the union's problem is to set $w(n_j,t) = \bar{C}e^{\hat{\omega}}$ where

$$\hat{\omega} = \ln b_r + \ln(\theta_n/(\theta_n - 1)) \tag{61}$$

if this leaves some workers unemployed and otherwise to set a higher level of wages consistent with full employment, $w(n_j,t) = \bar{C}e^{\omega}$, where ω is the log full employment wage defined in (36). In other words, the union sets a constant minimum wage which leaves a gap between the utility of the members who work and those who are rest unemployed. The minimum wage is time-invariant, although it will vary across labor markets depending on the elasticity of substitution θ_n. This is exactly the type of policy that we have analyzed in this paper; the analysis here simply provides a link between the minimum wage and the preference parameters b_r and θ_n.

According to this model, the economy would be perfectly competitive in the absence of unions. By monopolizing a labor market, a union can extract the monopoly rent. It does this by raising wages in order to restrict employment and output and hence raise the price of the good produced in the labor market. It achieves exactly the same outcome as would be attained by a monopoly producer facing a competitive labor market.[7] The model predicts that unions will be more successful at raising wages in labor markets producing goods that have poor substitution possibilities, θ_n close to 1.

Some observers have noted that, while unionization raises unemployment rates, the effects are mitigated if unions coordinate their activities (Stephen Nickell and Richard Layard, 1999). Our model suggests that this may because coordinated unions are able to internalize the impact of exploiting their monopoly power on other workers. The Pareto optimal allocation is achieved by dropping the minimum wage constraints, so a worker

[7]We do not analyze the interaction between a monopoly producer a monopoly union. In this case, setting a wage and allowing the firm to determine employment is generally inefficient. The two monopolists should agree on both a wage and a level of employment. Still, it seems likely that the equilibrium outcome will be a wage floor.

can work whenever $\omega = \ln b_r$ (see Alvarez and Shimer, 2011, Appendix B.2). Perhaps coordinated unions are able to avoid the incentive to restrict output in individual labor markets.

6 Conclusion

Our work analyzes the impact of unions on unemployment and wage dynamics in a model whose building blocks rely on the canonical work of Lucas and Prescott (1974). By imposing a minimum wage and seniority rule, unions cause rest unemployment, with recently laid-off workers intentionally staying within an under-performing labor market with the hope of regaining employment as the conditions improve. Hazard rates out of unemployment are thus steeply downward sloping, with many short jobless spells as well as few long ones, with an average unemployment duration that increases with the level of minimum wage. Unionized labor markets in our model feature wage compression — and in the extreme case where the minimum wage binds all the time, a constant wage rate irrespective of the evolution of sectorial productivity. Seniority rules, by backloading workers' payoff into the future, reduce the overall level of unemployment, relative to an alternative rule where jobs are allocated randomly to workers, thereby improving efficiency. While not the focus of our work, one could envision other job-worker allocation mechanisms, effectively assigning property rights to workers so as to alter their incentives to enter and exit labor markets, ultimately reducing further unemployment. Our theory and model predictions should be viewed as a starting point for the study of the influence of unions on labor market dynamics, as the increasing empirical evidence (see for instance Bhuller et al. (2022)) suggests that today's collective bargaining systems feature significant degrees of heterogeneity across OECD countries.

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Appendix

A.1 Proofs: Frictionless Model

A.1.1 Dynamics of Seniority

The dynamics of the measure of workers $\ell(t)$ in a given labor market can be derived from $d\log\ell(t) = (\theta-1)d\log x(t)$ using Itô's lemma:

$$d\ell(t) = \mu_\ell(t)dt + \sigma_\ell(t)dz(t),$$

where we have noted $\mu_\ell(t) = (\theta-1)\left(\mu_x + \frac{1}{2}(\theta-1)\sigma_x^2\right)\ell(t)$ and $\sigma_\ell(t) = (\theta-1)\sigma_x\ell(t)$. Assume that $s(t)$ satisfies the stochastic differential equation:

$$ds(t) = \mu_s(t)dt + \sigma_s(t)dz(t)$$

The postulated equilibrium condition (15) can then re-written:

$$\begin{aligned} d\left(s(t)\ell(t)\right) &= \left(\ell(t)\mu_s(t) + s(t)\mu_\ell(t) + \sigma_s(t)\sigma_\ell(t)\right)dt + \left(\ell(t)\sigma_s(t) + s(t)\sigma_\ell(t)\right)dz(t) \\ &= d\ell(t) + q\ell(t)\left(1-s(t)\right)dt \end{aligned}$$

Identifying the drift and volatility of the seniority $s(t)$ is then straightforward, after applying Itô's lemma:

$$\begin{aligned} \mu_s(t) &= (1-s)\left[1 + (\theta-1)\left(\mu_x - \frac{1}{2}(\theta-1)\sigma_x^2\right)\right] \\ \sigma_s(t) &= (1-s)(\theta-1)\sigma_x \end{aligned}$$

$\square$

A.1.2 Stationary Distribution of Productivity

Let $\tilde{g}(y)$ be the stationary distribution of log productivity $y = \log x$. Let us assume that the cumulative distribution function $\tilde{F}$ of new goods' log productivities admits a density

$\tilde{f}$, such that $\tilde{F}'(y) = \tilde{f}(y)$. $\tilde{g}(\cdot)$ solves the Kolmogorov forward equation:

$$\delta\tilde{g}(y) = -\mu_x\tilde{g}'(y) + \frac{1}{2}\sigma_x^2\tilde{g}''(y) + \delta\tilde{f}(y) \tag{A1}$$

The general solution of this second order ordinary differential equation takes the form:

$$C_1e^{\tilde{\eta}_1 y} + C_2e^{\tilde{\eta}_2 y},$$

where $\tilde{\eta}_1 < 0 < \tilde{\eta}_2$ and $\tilde{\eta}_1, \tilde{\eta}_2$ are the two real solutions of the quadratic equation:

$$\frac{1}{2}\sigma_x^2\tilde{\eta}^2 - \mu_x\tilde{\eta} - \delta = 0$$

One can easily verify that a particular solution to Equation (A1) can be computed as follows:

$$\alpha_1(y)e^{\tilde{\eta}_1 y} + \alpha_2(y)e^{\tilde{\eta}_2 y}$$

in which we have noted:

$$\begin{aligned}
\alpha_1(y) &= \frac{2\delta}{(\tilde{\eta}_2 - \tilde{\eta}_1)\sigma_x^2}\int_K^y e^{-\tilde{\eta}_1 u}\tilde{f}(u)du \\
\alpha_2(y) &= \frac{-2\delta}{(\tilde{\eta}_2 - \tilde{\eta}_1)\sigma_x^2}\int_K^y e^{-\tilde{\eta}_2 u}\tilde{f}(u)du
\end{aligned}$$

The complete solution of Equation (A1) thus takes the form

$$\tilde{g}(y) = (C_1 + \alpha_1(y))\,e^{\tilde{\eta}_1 y} + (C_2 + \alpha_2(y))\,e^{\tilde{\eta}_2 y},$$

where C_1, C_2, K verify $\int_{-\infty}^{+\infty}\tilde{g}(u)du = 1$, $\lim_{u\to-\infty}\tilde{g}(u) = 0$ and $\lim_{u\to+\infty}\tilde{g}(u) = 0$. $\square$

A.1.3 Solving the Model

Using the laws of motion for $s(t)$ in our conjectured equilibrium, the worker's value function v satisfies the following Hamilton-Jacobi-Bellman equation:

$$\rho v(s) = R(s) + \lambda \left(\frac{w^*}{\rho C} - v(s) \right) + (1-s) \left[q + (\theta - 1) \left(\mu_x - \frac{1}{2}(\theta - 1)\sigma_x^2 \right) \right] v'(s) + \frac{1}{2}(1-s)^2(\theta-1)^2\sigma_x^2 v''(s) \quad \text{(A2)}$$

for all $s \in (0, 1)$. Equation (A2) assumes that $v(s)$ is twice continuously differentiable at s where $R(s)$ is continuous, although it is only once differentiable at $s = \hat{s}$. We then focus on boundary conditions. The value matching condition states that workers with zero seniority are indifferent about participating in the market and going to a competitive market,

$$v(0) = \frac{w^*}{\rho C}.$$

The smooth pasting condition states that the marginal value of seniority is zero at low seniority,

$$v'(0) = 0.$$

We establish the latter condition in Appendix A.1.4. Finally, note that seniority $s = 1$, aside from the risk of the household member suffering an exogenous quit or the related labor market shutting down, is an absorbing state. This means that

$$\rho v(1) = \frac{\hat{w}}{C} + \lambda \left(\frac{w^*}{\rho C} - v(1) \right),$$

which ensures that the marginal value of seniority is bounded at $s = 1$.

Equation (A2) is a second order ordinary differential equation, with a period return function that is either b_r or $\hat{w}/C$. Moreover, we also look for the endogenous threshold $\hat{s}$, above which the worker is employed and below which the worker is rest unemployed. We thus need 5 equations to solve this second order ordinary differential equations on two sub-intervals $s \in [0, \hat{s}) \cup (\hat{s}, 1]$. Two equations will come from value matching and smooth pasting conditions at $s = 0$, one equation will come from the fact that the value function must be finite at $s = 1$, and the last two equations will come from the continuous differentiability of v as $s = \hat{s}$. This yields:

$$v(s) = \begin{cases} \dfrac{b_r}{\rho + \lambda} + \dfrac{\lambda w^*/C}{(\rho+\lambda)\rho} + \displaystyle\sum_{i=1}^{2} \underline{c}_i (1-s)^{-\eta_i} & \text{if } s < \hat{s} \\ \dfrac{\hat{w}/C}{\rho + \lambda} + \dfrac{\lambda w^*/C}{(\rho+\lambda)\rho} + \displaystyle\sum_{i=1}^{2} \hat{c}_i (1-s)^{-\eta_i} & \text{if } s \geq \hat{s}, \end{cases} \quad \text{(A3)}$$

where the exponents $\{\eta_i\}_{i=1,2}$ are the two roots of Equation (17). Note that $\eta_1 < 0$ and

$\eta_2 > 1$, the latter condition being ensured by Equation (6). The threshold for working $\hat{s}$, and hence the unemployment rate in the market, is given by Equation (18). Combining equations (12) and (18) yields Equation (19) for $\bar{w}$. Since $\hat{w} > w^*$, $\theta > 1$, and $\eta_2 > 1$, this implies $\bar{w} < \hat{w}$. The constants $\underline{c}_i$ and $\hat{c}_i$ satisfy

$$\underline{c}_1 = \left(\frac{w^*/C - b_r}{\rho + \lambda}\right) \frac{\eta_2}{\eta_2 - \eta_1} > 0, \quad \underline{c}_2 = -\left(\frac{w^*/C - b_r}{\rho + \lambda}\right) \frac{\eta_1}{\eta_2 - \eta_1} > 0,$$
$$\hat{c}_1 = -\left(\frac{w^*/C - b_r}{\rho + \lambda}\right) \frac{\eta_2}{\eta_2 - \eta_1} \left(\left(\frac{\hat{w} - b_r C}{w^* - b_r C}\right)^{\frac{\eta_2 - \eta_1}{\eta_2}} - 1\right) < 0, \quad \hat{c}_2 = 0.$$

The general form of the value function in (A3) is the unique solution to the differential (A2) at all points $s \in [0, \hat{s}) \cup (\hat{s}, 1]$. The constants $\underline{c}_1$ and $\underline{c}_2$ are pinned down by the value-matching and smooth-pasting conditions. The restriction $\hat{c}_2 = 0$ is required to be sure that the value function stays bounded as seniority converges to 1. Finally, the choice of $\hat{c}_1$ and $\hat{s}$ is determined by the requirement that the value function is everywhere once differentiable, $\lim_{s \to \hat{s}_+} v(s) = \lim_{s \to \hat{s}_-} v(s)$ and $\lim_{s \to \hat{s}_+} v'(s) = \lim_{s \to \hat{s}_-} v'(s)$.

It is straightforward to verify algebraically that the value function is increasing in s. First, note that on the interval $[0, \hat{s}]$, v is strictly convex in s. Since $v'(0) = 0$, $v'(s)$ is increasing for $s < \hat{s}$[8]. At values of $s > \hat{s}$, $v'(s)$ is positive because $\hat{c}_1 < 0$ and $\eta_1 < 0$. This confirms that workers exit their labor market voluntarily only when their seniority falls to 0. □

A.1.4 Smooth Pasting Condition

Note $\mu_s(s) = (1-s)\left[q + (\theta - 1)\left(\mu_x - \frac{1}{2}(\theta - 1)\sigma_x^2\right)\right]$ and $\sigma_s(s) = (1-s)(\theta-1)\sigma_x$. We consider a discrete time, discrete state space Markov chain approximation of the stochastic process $\{s(t)\}_{t \geq 0}$. Our approximate Markov Chain is parametrized by the small step $h > 0$. Take some arbitrary $\epsilon > 0$. Note $Q^h(s) \equiv \sigma_s^2(s) + h|\mu_s(s)| + \epsilon h$, and let $\Delta t^h \equiv h^2/Q^h(s)$ be a small time step. Note that $\inf_{s \in [0,1]} Q^h(s) > 0$, meaning that Δt^h is well defined for all $s \in [0, 1]$. The discount factor is $1 - \rho \Delta t^h$, and the exogenous exit probability is $\lambda \Delta t^h$. s lies on the grid $\{0, h, \ldots, (N_h - 1)h, N_h h\}$, with $N_h \equiv 1/h$. Each time period, s increases by h with probability $p_u^h(s)$, decreases by h with probability

[8] It is actually strictly increasing on $(0, \hat{s}]$

$p_d^h(s)$, and stays constant with probability $p_f^h(s)$ where those probabilities satisfy:

$$\begin{aligned}
p_u^h(s) &:= \frac{\frac{1}{2}\sigma_s^2(s) + h\max(0, \mu_s(s))}{Q^h(s)} \\
p_d^h(s) &:= \frac{\frac{1}{2}\sigma_s^2(s) + h\max(0, -\mu_s(s))}{Q^h(s)} \\
p_c^h(s) &:= \frac{\epsilon h}{Q^h(s)}
\end{aligned}$$

Note that those probabilities are greater than zero and add up to 1. The Markov chain approximation constructed is consistent: $E[\Delta s|s] = \mu_s(s)\Delta t^h$ and $\text{var}[\Delta s|s] = \sigma_s^2(s)\Delta t^h + o(\Delta t^h)$. Moreover, $\lim_{h\to 0}\Delta t^h = 0$. At seniority level 0, between t and $t+\Delta t^h$, a worker's value can be written as follows:

$$v(0) = b_r\Delta t^h + (1-\rho\Delta t^h)\left[\frac{b_i}{\rho}\lambda\Delta t^h + (1-\lambda\Delta t^h)\left(p_u^h(0)v(0+h) + p_f^h(0)v(0) + p_d^h(0)\frac{b_i}{\rho}\right)\right]$$

Taking a first order Taylor expansion of v around $s=0$ and using $v(0) = b_i/\rho$, the terms of order zero cancel out, while the terms of order h only cancel out if $v'(0) = 0$, which is the smooth pasting condition we were looking for. □

A.1.5 Closing the Model

To determine aggregate consumption C, we can use the fact that in equilibrium, the consumption of good j is $c(j,t) = Ax(j,t)\ell(j,t)$. Using equations (11) and (19), we obtain an expression for $c(j,t)$ as a function of the minimum wage and of $w^* = b_iC$ (the wage in unconstrained labor markets):

$$c(j,t) = C\frac{A^{\theta-1}}{\max(w^*, \hat{w}(j))^{\theta}}\left(\frac{\max(w^*, \hat{w}(j)) - b_rC}{w^* - b_rC}\right)^{1/\eta_2} x(j,t)^{\theta}$$

Using Equation (3) gives an implicit equation that C needs to satisfy in equilibrium:

$$1 = \int_0^1 \frac{A^{\frac{(\theta-1)^2}{\theta}}}{\max(w^*, \hat{w}(j))^{\theta-1}}\left(\frac{\max(w^*, \hat{w}(j)) - b_rC}{w^* - b_rC}\right)^{\frac{\theta-1}{\eta_2\theta}} x(j,t)^{\theta-1}dj$$

In Appendix A.1.2, we calculated an analyical expression for the stationary distribution $\tilde{g}(\cdot)$ of log productivity $\log x$ as a function of δ, μ_x, σ_x and the new firms' productivity distribution $F(\cdot)$. Note that $\tilde{g}$ does not depend on the parameters ρ, q and θ. Noting $\mu(j)$ the mass of labor markets with minimum wage $\hat{w}(j)$, aggregate consumption C is solution to the following:

$$\frac{1}{A^{\frac{(\theta-1)^2}{\theta}}\left(\int_{\infty}^{+\infty} e^{(\theta-1)u}\tilde{g}(u)du\right)} = \left(\sum_{j:\hat{w}(j)>b_iC} \frac{\mu(j)}{\hat{w}(j)^{\theta-1}}\left(\frac{\hat{w}(j)/C - b_r}{b_i - b_r}\right)^{\frac{\theta-1}{\eta_2\theta}} + \left(1 - \sum_{j:\hat{w}(j)>b_iC}\mu(j)\right)\frac{1}{(b_iC)^{\theta-1}}\right)$$

A.2 Proofs: Full Model

A.2.1 Value Function Derivation

A worker in state (ω_0, s_0) in a market characterized by a log minimum wage $\hat{\omega}$ and boundaries $(\underline{\omega}, \bar{\omega})$, has a value function that can be expressed as follows:

$$v(\omega_0, s_0; \hat{\omega}, \underline{\omega}, \bar{\omega}) = \mathbb{E}_{\omega_0, s_0}\left[\int_0^\tau e^{-\rho t}R(\omega(t), s(t))dt + e^{-\rho\tau}v\left(\omega(\tau), s(\tau); \hat{\omega}, \underline{\omega}, \bar{\omega}\right)\right]$$

In the above, $\tau \equiv \tau(\underline{\omega}, 0) \wedge \tau_\lambda$. The stopping time $\tau(\underline{\omega}, 0) = \inf\{t : (\omega(t), s(t)) = (\underline{\omega}, 0)\}$ is the first time the worker reaches state $(\underline{\omega}, 0)$. The stopping time τ_λ is an exponentially distributed random variable, with arrival intensity λ. We can then use the law of iterated expectations and condition on the realization of the Brownian path (and therefore the realization of the stopping time $\tau(\underline{\omega}, 0)$):

$$v(\omega_0, s_0; \hat{\omega}, \underline{\omega}, \bar{\omega}) = \mathbb{E}_{\omega_0, s_0}\left[\mathbb{E}_{\omega_0, s_0}\left[\int_0^\tau e^{-\rho t}R(\omega(t), s(t))dt + e^{-\rho\tau}v\left(\omega(\tau), s(\tau); \hat{\omega}, \underline{\omega}, \bar{\omega}\right)\middle|\tau(\underline{\omega}, 0)\right]\right]$$

We note that the conditional expectation can be expressed as follows:

$$\mathbb{E}^{\omega_0, s_0}\left[\int_0^\tau e^{-\rho t}R(\omega(t), s(t))dt + e^{-\rho\tau}v\left(\omega(\tau), s(\tau); \hat{\omega}, \underline{\omega}, \bar{\omega}\right)\middle|\tau(\underline{\omega}, 0)\right]$$

$$= e^{-\lambda\tau(\underline{\omega},0)} \times \left(\int_0^{\tau(\underline{\omega},0)} e^{-\rho t} R(\omega(t), s(t)) dt + e^{-\rho\tau(\underline{\omega},0)} \underline{v} \right) + \int_0^{\tau(\underline{\omega},0)} \lambda e^{-\lambda z} \left(\int_0^{z} e^{-\rho t} R(\omega(t), s(t)) dt + e^{-\rho z} \underline{v} \right) dz$$

Integration by part of the second integral above leads us to:

$$\mathbb{E}^{\omega_0, s_0} \left[\int_0^{\tau} e^{-\rho t} R(\omega(t), s(t)) dt + e^{-\rho\tau} v\left(\omega(\tau), s(\tau); \hat{\omega}, \underline{\omega}, \bar{\omega}\right) \middle| \tau(\underline{\omega}, 0) \right] = \int_0^{\tau(\underline{\omega},0)} e^{-(\rho+\lambda)t} \left(R(\omega(t), s(t)) + \lambda \underline{v} \right) dt + e^{-(\rho+\lambda)\tau(\underline{\omega},0)} \underline{v}$$

We thus obtain the desired result:

$$v(\omega_0, s_0; \hat{\omega}, \underline{\omega}, \bar{\omega}) = \mathbb{E}^{\omega_0, s_0} \left[\int_0^{\tau(\underline{\omega},0)} e^{-(\rho+\lambda)t} \big(R(\omega(t), s(t)) + \lambda \underline{v} \big) dt + e^{-(\rho+\lambda)\tau(\underline{\omega},0)} \underline{v} \right]$$

$\square$

A.2.2 Differentiability of the Value Function

Note that for $\omega \in (\underline{\omega}, \bar{\omega})$, a worker's seniority is constant; although some workers exit the market exogenously, they are drawn uniformly from the population of workers. Now let $\tau(\underline{\omega}) = \inf\{t : \omega(t) = \underline{\omega}\}$ and $\tau(\bar{\omega}) = \inf\{t : \omega(t) = \bar{\omega}\}$ denote the stopping times when ω first hits $\underline{\omega}$ and $\bar{\omega}$, infinite if it hits the other boundary first. Then we can rewrite (42) as

$$v(\omega_0, s_0; \hat{\omega}, \underline{\omega}, \bar{\omega}) = \mathbb{E}_{\omega_0} \Bigg[\int_0^{\tau(\underline{\omega}) \wedge \tau(\bar{\omega})} e^{-(\rho+\lambda)t} \big(R(\omega(t), s_0) + \lambda \underline{v} \big) dt + 1_{\{\tau(\underline{\omega}) < \tau(\bar{\omega})\}} v(\underline{\omega}, s_0) e^{-(\rho+\lambda)\tau(\underline{\omega})} + 1_{\{\tau(\underline{\omega}) > \tau(\bar{\omega})\}} v(\bar{\omega}, s_0) e^{-(\rho+\lambda)\tau(\bar{\omega})} \Bigg].$$

This can be re-written:

$$v(\omega_0, s_0; \hat{\omega}, \underline{\omega}, \bar{\omega}) = \int_{\underline{\omega}}^{\bar{\omega}} \left(R(\omega, s_0) + \lambda \underline{v}\right) \pi\left(\omega, \omega_0; \underline{\omega}, \bar{\omega}\right) d\omega + \psi\left(\omega_0; \underline{\omega}, \bar{\omega}\right) v\left(\underline{\omega}, s_0\right) + \Psi\left(\omega_0; \underline{\omega}, \bar{\omega}\right) v\left(\bar{\omega}, s_0\right)$$

In the above, $\pi\left(\omega, \omega_0; \underline{\omega}, \bar{\omega}\right)$ is the discounted local time up to the stopping time $\tau \equiv \tau(\underline{\omega}) \wedge \tau(\bar{\omega})$, and the functions ψ, Ψ are defined as:

$$\psi\left(\omega_0; \underline{\omega}, \bar{\omega}\right) = \mathbb{E}\left[e^{-(\rho+\lambda)\tau} | \tau = \tau(\underline{\omega}), \omega(0) = \omega_0\right] \Pr\left(\tau = \tau(\underline{\omega}) | \omega(0) = \omega_0\right) \tag{A4}$$

$$\Psi\left(\omega_0; \underline{\omega}, \bar{\omega}\right) = \mathbb{E}\left[e^{-(\rho+\lambda)\tau} | \tau = \tau(\bar{\omega}), \omega(0) = \omega_0\right] \Pr\left(\tau = \tau(\bar{\omega}) | \omega(0) = \omega_0\right) \tag{A5}$$

And $\tau \equiv \tau(\underline{\omega}) \wedge \tau(\bar{\omega})$. In proposition 5.3, Stokey (2008) shows that:

$$\psi\left(\omega_0; \underline{\omega}, \bar{\omega}\right) = \frac{e^{\eta_1(\omega_0 - \bar{\omega})} - e^{\eta_2(\omega_0 - \bar{\omega})}}{e^{\eta_1(\underline{\omega} - \bar{\omega})} - e^{\eta_2(\underline{\omega} - \bar{\omega})}}$$
$$\Psi\left(\omega_0; \underline{\omega}, \bar{\omega}\right) = \frac{e^{\eta_1(\omega_0 - \underline{\omega})} - e^{\eta_2(\omega_0 - \underline{\omega})}}{e^{\eta_1(\bar{\omega} - \underline{\omega})} - e^{\eta_2(\bar{\omega} - \underline{\omega})}}$$

In exercise 5.4, Stokey (2008) shows that:

$$\pi\left(\omega, \omega_0; \underline{\omega}, \bar{\omega}\right) = \begin{cases} \frac{1}{\left(\mu^2 + 2(\rho+\lambda)\sigma^2\right)^{1/2}} \left[e^{\eta_1(\omega_0 - \omega)} - \Psi(\omega_0) e^{\eta_1(\bar{\omega} - \omega)} - \psi(\omega_0) e^{\eta_2(\underline{\omega} - \omega)}\right] & \text{if } \underline{\omega} \leq \omega \leq \omega_0 \\ \frac{1}{\left(\mu^2 + 2(\rho+\lambda)\sigma^2\right)^{1/2}} \left[e^{\eta_2(\omega_0 - \omega)} - \Psi(\omega_0) e^{\eta_1(\bar{\omega} - \omega)} - \psi(\omega_0) e^{\eta_2(\underline{\omega} - \omega)}\right] & \text{if } \omega_0 \leq \omega \leq \bar{\omega}, \end{cases}$$

Since π is everywhere continuous and is continuously differentiable except at ω_0, differentiating the previous expression gives

$$\frac{\partial v(\omega_0, s_0; \hat{\omega}, \underline{\omega}, \bar{\omega})}{\partial \omega_0} = \int_{\underline{\omega}}^{\bar{\omega}} \left(R(\omega(t), s_0) + \lambda \underline{v}\right) \frac{\partial \pi(\omega, \omega_0; \underline{\omega}, \bar{\omega})}{\partial \omega_0} dt + \left(\lim_{\omega \uparrow \omega_0} R(\omega, s_0) - \lim_{\omega \downarrow \omega_0} R(\omega, s_0)\right) \pi(\omega_0, \omega_0; \underline{\omega}, \bar{\omega}) + v(\underline{\omega}, s_0) \frac{\partial \psi(\omega_0; \underline{\omega}, \bar{\omega})}{\partial \omega_0} + v(\bar{\omega}, s_0) \frac{\partial \Psi(\omega_0; \underline{\omega}, \bar{\omega})}{\partial \omega_0}$$

This in turn is continuously differentiable if $R(\omega, s_0)$ is continuous at $\omega = \omega_0$, i.e. if $s_0 \neq 1 - e^{\theta(\omega - \hat{\omega})}$. $\square$

A.2.3 Discrete Time, Discrete State Space Full Model

We consider a discrete time, discrete state space Markov chain approximation of the stochastic process $\{\omega(t)\}_{t\geq 0}$. Our approximate Markov chain is parametrized by $h > 0$. Let $\Delta t^h = h^2/\sigma^2$ be a small time period, and let $\Delta p^h = \frac{\mu}{\sigma^2}h$. The discount factor is $1-\rho\Delta t^h$, and the exogenous exit probability is $\lambda\Delta t^h$. ω lies on the grid $\{\underline{\omega}, \underline{\omega}+h, \ldots, \underline{\omega}+(n-1)h, \underline{\omega}+nh \equiv \bar{\omega}\}$. Between t and $t+\Delta t^h$, when $\underline{\omega} < \omega < \bar{\omega}$, ω increases by h with probability $p_u^h = \frac{1}{2}(1+\Delta p^h)$ and decreases by h with probability $p_d^h = \frac{1}{2}(1-\Delta p^h)$, while the seniority s stays constant. The Markov chain approximation constructed is consistent: $E[\Delta\omega] = \mu\Delta t^h$ and $\text{var}[\Delta\omega] = \sigma^2\Delta t^h + o(\Delta t^h)$. Moreover, $\lim_{h\to 0}\Delta t^h = 0$.

If $\omega = \underline{\omega}$, it increases to $\underline{\omega}+h$ with probability p_u^h. Otherwise, if there is a negative shock, ω is unchanged and all workers with seniority $s < 1-e^{-\theta h}$ exit the labor market. This ensures that log employment falls by θh, which according to (36) is enough to leave the log full employment wage constant. The seniority of all other workers changes as well, falling from s to

$$s - \Delta s^h = \frac{s-1+e^{-\theta h}}{e^{-\theta h}}. \tag{A6}$$

and $\Delta s^h = (1-s)(e^{\theta h}-1) = (1-s)\theta h + o(h)$ when $h \to 0$.

Conversely, if $\omega = \bar{\omega}$, a negative shock reduces it to $\bar{\omega}-h$ with probability p_d^h. Otherwise there is a positive shock. ω is unchanged, but log employment rises by θh in order to leave the log full employment wage constant. The seniority of all workers rises from s to

$$s + \Delta s^h = \frac{s-1+e^{\theta h}}{e^{\theta h}}. \tag{A7}$$

and $\Delta s^h = (1-s)(1-e^{-\theta h}) = (1-s)\theta h + o(h)$ when $h \to 0$.

First take ω on the interior of the grid and an arbitrary s. The Bellman equation implies

$$\begin{aligned} v(\omega,s) = R(\omega,s)\Delta t^h + (1-\rho\Delta t^h)\Big(&\lambda\Delta t^h \underline{v} \\ &+ (1-\lambda\Delta t^h)\big(\tfrac{1}{2}(1+\Delta p^h)v(\omega+h,s) + \tfrac{1}{2}(1-\Delta p^h)v(\omega-h,s)\big)\Big). \end{aligned}$$

Take a second order Taylor expansion to $v(\omega \pm h, s)$ around $v(\omega,s)$ and simplify:

$$\begin{aligned} v(\omega,s) = R(\omega,s)\frac{h^2}{\sigma^2} + (1-\rho\frac{h^2}{\sigma^2})\Big(&\lambda\frac{h^2}{\sigma^2}\underline{v} \\ &+ (1-\lambda\frac{h^2}{\sigma^2})\big(v(\omega,s) + v_\omega(\omega,s)\frac{\mu h^2}{\sigma^2} + \frac{1}{2}v_{\omega\omega}(\omega,s)h^2\big)\Big) + o(h^2) \end{aligned}$$

The zero order terms cancel out, while there are no first order term. Dividing by $h^2 > 0$ and taking the limit as $h \to 0$ gives (47).

Now consider $\omega = \underline{\omega}$. For $s > 1 - e^{-\theta h}$, the Bellman equation solves

$$v(\underline{\omega}, s) = R(\underline{\omega}, s)\Delta t^h + (1 - \rho\Delta t^h)\Big(\lambda\Delta t^h \underline{v} \\ + (1 - \lambda\Delta t^h)\big(\tfrac{1}{2}(1 + \Delta p^h)v(\underline{\omega} + h, s) + \tfrac{1}{2}(1 - \Delta p^h)v(\underline{\omega}, s - \Delta s^h)\big)\Big).$$

Now take a first order Taylor expansion of v around $(\underline{\omega}, s)$; higher order terms would disappear from the expression. We obtain

$$v(\underline{\omega}, s) = R(\underline{\omega}, s)\frac{h^2}{\sigma^2} + (1 - \rho\frac{h^2}{\sigma^2})\Big(\lambda\frac{h^2}{\sigma^2}\underline{v} \\ + (1 - \lambda\frac{h^2}{\sigma^2})\big(v(\underline{\omega}, s) + \frac{1}{2}(1 + \frac{\mu h}{\sigma^2})v_\omega(\underline{\omega}, s)h - \frac{1}{2}(1 - \frac{\mu h}{\sigma^2})v_s(\underline{\omega}, s)(1 - s)\theta h\big)\Big) + o(h)$$

The zero order terms cancel out. Dividing by $h > 0$ and taking the limit as $h \to 0$ gives (48). The derivation of (49) is almost identical and hence omitted.

Now take $\omega = \underline{\omega}$ and $s \leq 1 - e^{-\theta h}$. In this case, the Bellman equation solves

$$v(\underline{\omega}, s) = R(\underline{\omega}, s)\Delta t^h + (1 - \rho\Delta t^h)\Big(\lambda\Delta t^h \underline{v} + (1 - \lambda\Delta t^h)\big(\tfrac{1}{2}(1 + \Delta p^h)v(\underline{\omega} + h, s) + \tfrac{1}{2}(1 - \Delta p^h)\underline{v}\big)\Big).$$

Taking a first order Taylor expansion of v around $(\underline{\omega}, 0)$ and using $v(\underline{\omega}, 0) = \underline{v}$ gives

$$\underline{v} + v_s(\underline{\omega}, 0)s = R(\underline{\omega}, s)\frac{h^2}{\sigma^2} + (1 - \rho\frac{h^2}{\sigma^2})\Big(\lambda\frac{h^2}{\sigma^2}\underline{v} \\ + (1 - \lambda\frac{h^2}{\sigma^2})\Big(\underline{v} + \tfrac{1}{2}(1 + \frac{\mu h}{\sigma^2})\big(v_\omega(\underline{\omega}, 0)h + v_s(\underline{\omega}, 0)s\big)\Big)\Big) + o(h)$$

The zero order terms cancel out. Dividing by $h > 0$, taking the limit as $h \to 0$ and combining with (48) gives (50).

Finally we handle the case of $s = 1$. Since the seniority of such a worker never changes (see equations (A6) and A7), we have

$$v(\underline{\omega}, 1) = R(\underline{\omega}, 1)\Delta t^h + (1 - \rho\Delta t^h)\Big(\lambda\Delta t^h \underline{v} \\ + (1 - \lambda\Delta t^h)\big(\tfrac{1}{2}(1 + \Delta p^h)v(\underline{\omega} + h, 1) + \tfrac{1}{2}(1 - \Delta p^h)v(\underline{\omega}, 1)\big)\Big).$$

Take a first order Taylor expansion of v around $(\underline{\omega}, 1)$:

$$v(\underline{\omega},1) = R(\underline{\omega},1)\frac{h^2}{\sigma^2} + (1-\rho\frac{h^2}{\sigma^2})\Big(\lambda\frac{h^2}{\sigma^2}\underline{v} + (1-\lambda\frac{h^2}{\sigma^2})\big(v(\underline{\omega},1) + \tfrac{1}{2}(1+\frac{\mu h}{\sigma^2})v_\omega(\underline{\omega},1)h\big)\Big) + o(h)$$

The zero order terms cancel out. Dividing by $h > 0$, taking the limit as $h \to 0$ gives $v_\omega(\underline{\omega},1) = 0$. A similar logic at $(\bar{\omega},1)$ gives $v_\omega(\bar{\omega},1) = 0$, establishing (51). $\square$

A.2.4 Expression for the Value Function given Thresholds

A.2.4.1 High Minimum Wage

We first tackle the case where $\hat{\omega} \geq \bar{\omega}$. We claim that the value function satisfies

$$v(\omega,s) = \begin{cases} \dfrac{e^{\hat{\omega}} + \lambda\underline{v}}{\rho+\lambda} + \hat{c}_1(s)e^{\eta_1(\omega-\hat{\omega})} + \hat{c}_2(s)e^{\eta_2(\omega-\hat{\omega})} \text{ if } s \geq 1 - e^{\theta(\omega-\hat{\omega})} \\ \dfrac{b_r + \lambda\underline{v}}{\rho+\lambda} + \underline{c}_1(s)e^{\eta_1(\omega-\hat{\omega})} + \underline{c}_2(s)e^{\eta_2(\omega-\hat{\omega})} \text{ if } s < 1 - e^{\theta(\omega-\hat{\omega})}, \end{cases} \tag{A8}$$

where $\eta_1 < 0 < 1 < \eta_2$ are the roots of the characteristic (40), and the univariate functions of integration satisfy

$$\hat{c}_1(s) = -\left(\frac{\eta_2}{\eta_2-\eta_1}\right)\left(\frac{e^{\hat{\omega}} - b_r}{\rho+\lambda}\right)(1 - e^{-(\eta_2-\eta_1)(\hat{\omega}-\underline{\omega})})(1-s)^{-\eta_1/\theta} \tag{A9}$$

$$\hat{c}_2(s) = 0 \tag{A10}$$

$$\underline{c}_1(s) = \hat{c}_1(s) + \left(\frac{\eta_2}{\eta_2-\eta_1}\right)\left(\frac{e^{\hat{\omega}} - b_r}{\rho+\lambda}\right)(1-s)^{-\eta_1/\theta} \tag{A11}$$

$$\underline{c}_2(s) = \hat{c}_2(s) + \left(\frac{-\eta_1}{\eta_2-\eta_1}\right)\left(\frac{e^{\hat{\omega}} - b_r}{\rho+\lambda}\right)(1-s)^{-\eta_2/\theta}, \tag{A12}$$

where we leave the expressions in a convenient form.

To prove this, note first that (A8) is the general solution to (47). All that remains is to characterize the four functions of integration. The condition that the value function is continuously differentiable even at the boundary between work and rest unemployment, i.e. points of the form $(\omega, 1 - e^{\theta(\omega-\hat{\omega})})$, yields equations (A11) and (A12). Equation (48) and (49) then reduce to $\hat{c}_i(s)\eta_i = \hat{c}_i'(s)(1-\theta)s$ for $i = 1,2$, or equivalently $\hat{c}_i(s) = c_i(1-s)^{-\eta_i/\theta}$.

To pin down the two constants c_1 and c_2, we use two more boundary conditions. The value function needs to have finite value at $s = 1$, which implies $c_2 = 0$ and (A10). Finally, use (50) to pin down the last constant c_1, yielding (A9). Note that equations (51) are automatically satisfied.

Finally, note that we can perform a change in variable $y \equiv \frac{e^{\theta(\omega-\hat{\omega})}}{1-s}$, and express the value function as a function of y:

$$v(y) = \begin{cases} \dfrac{e^{\hat{\omega}} + \lambda \underline{v}}{\rho + \lambda} + \hat{c}_1(0) y^{\eta_1/\theta} \text{ if } y \geq 1 \\ \dfrac{b_r + \lambda \underline{v}}{\rho + \lambda} + \underline{c}_1(0) y^{\eta_1/\theta} + \underline{c}_2(0) y^{\eta_2/\theta} \text{ if } y < 1 \end{cases} \tag{A13}$$

In this particular environment, the worker only cares about his seniority relative to some rescaled version of the log full-employment wage ω. If such renormalized state variable y is greater than 1, the worker is employed at the minimum wage $\hat{w}$, while if y is less than 1, the worker is rest-unemployed and receives b_r in utils. Note that y_t evolves as follows:

$$\frac{dy_t}{y_t} = \left[\theta\mu + \frac{1}{2}\theta^2\sigma^2\right] dt + \theta\sigma dz_t$$

Those dynamics need to be compared to the dynamics of the state variable $(1 - s_t)^{-1}$ in our frictionless model of section 2:

$$\frac{d\left(1 - s_t\right)^{-1}}{\left(1 - s_t\right)^{-1}} = \left[q + (\theta - 1)\mu_x + \frac{1}{2}(\theta - 1)^2\sigma_x^2\right] dt + (\theta - 1)\sigma_x dz_t$$

Since μ and σ depend on μ_x, σ_x according to (39), it is immediate to notice that the drift and diffusion coefficients of $(1 - s_t)^{-1}$ in our frictionless model are identical to those of y_t.

We end this section by noting that in the above and going forward, in the case of a high minimum wage, we assume $\hat{\omega} = \max\left(\log(\hat{w}/C), \log b_r\right) = \log(\hat{w}/C) > \log b_r$, since otherwise, workers receive b_r utils irrespective of their seniority or the level of ω, meaning that the value function would be constant and equal to $\frac{b_r + \lambda \underline{v}}{\rho + \lambda}$. $\square$

A.2.4.2 Moderate Minimum Wage

Next we turn to the case where $\bar{\omega} > \hat{\omega} \geq \underline{\omega}$. The basic approach is similar. We claim that the value function satisfies

$$v(\omega, s) = \begin{cases} \dfrac{e^{\omega}}{\rho + \lambda - \mu - \frac{1}{2}\sigma^2} + \dfrac{\lambda \underline{v}}{\rho + \lambda} + \bar{c}_1(s) e^{\eta_1(\omega - \hat{\omega})} + \bar{c}_2(s) e^{\eta_2(\omega - \hat{\omega})} \text{ if } \omega \geq \hat{\omega} \\ \dfrac{e^{\hat{\omega}} + \lambda \underline{v}}{\rho + \lambda} + \hat{c}_1(s) e^{\eta_1(\omega - \hat{\omega})} + \hat{c}_2(s) e^{\eta_2(\omega - \hat{\omega})} \text{ if } \omega < \hat{\omega} \text{ and } s \geq 1 - e^{\theta(\omega - \hat{\omega})} \\ \dfrac{b_r + \lambda \underline{v}}{\rho + \lambda} + \underline{c}_1(s) e^{\eta_1(\omega - \hat{\omega})} + \underline{c}_2(s) e^{\eta_2(\omega - \hat{\omega})} \text{ if } \omega < \hat{\omega} \text{ and } s < 1 - e^{\theta(\omega - \hat{\omega})}, \end{cases} \tag{A14}$$

where $\eta_1 < 0 < 1 < \eta_2$ solve (40). The functions $\underline{c}_i, \hat{c}_i, \bar{c}_i$ are as follows:

$$\hat{c}_1(s) = c_1(1-s)^{-\eta_1/\theta} + \zeta_1 \tag{A15}$$

$$\hat{c}_2(s) = \zeta_2 \tag{A16}$$

$$\bar{c}_1(s) = \hat{c}_1(s) + a_1 e^{\eta_1 \hat{\omega}} \tag{A17}$$

$$\bar{c}_2(s) = \hat{c}_2(s) + a_2 e^{\eta_2 \hat{\omega}} \tag{A18}$$

$$\underline{c}_1(s) = \hat{c}_1(s) + \left(\frac{\eta_2}{\eta_2 - \eta_1}\right)\left(\frac{e^{\hat{\omega}} - b_r}{\rho + \lambda}\right)(1-s)^{-\eta_1/\theta} \tag{A19}$$

$$\underline{c}_2(s) = \hat{c}_2(s) + \left(\frac{-\eta_1}{\eta_2 - \eta_1}\right)\left(\frac{e^{\hat{\omega}} - b_r}{\rho + \lambda}\right)(1-s)^{-\eta_2/\theta} \tag{A20}$$

The constants c_1 and ζ_1, ζ_2 are equal to:

$$c_1 = -\left(\frac{\eta_2}{\eta_2 - \eta_1}\right)\left(\frac{e^{\hat{\omega}} - b_r}{\rho + \lambda}\right)\left(1 - e^{-(\eta_2 - \eta_1)(\hat{\omega} - \underline{\omega})}\right) \leq 0 \tag{A21}$$

$$\zeta_1 = \frac{e^{\eta_1 \hat{\omega} + \eta_2 \underline{\omega}}\left(\frac{e^{\bar{\omega}}}{\rho + \lambda - \mu - \frac{1}{2}\sigma^2} + \eta_1 e^{\eta_1 \bar{\omega}} a_1 + \eta_2 e^{\eta_2 \bar{\omega}} a_2\right)}{\eta_1\left(e^{\eta_1 \underline{\omega} + \eta_2 \bar{\omega}} - e^{\eta_1 \bar{\omega} + \eta_2 \underline{\omega}}\right)} \tag{A22}$$

$$\zeta_2 = -\frac{e^{\eta_1 \underline{\omega} + \eta_2 \hat{\omega}}\left(\frac{e^{\bar{\omega}}}{\rho + \lambda - \mu - \frac{1}{2}\sigma^2} + \eta_1 e^{\eta_1 \bar{\omega}} a_1 + \eta_2 e^{\eta_2 \bar{\omega}} a_2\right)}{\eta_2\left(e^{\eta_1 \underline{\omega} + \eta_2 \bar{\omega}} - e^{\eta_1 \bar{\omega} + \eta_2 \underline{\omega}}\right)} \tag{A23}$$

Note that $\zeta_2 = -\frac{\eta_1}{\eta_2} e^{(\eta_2-\eta_1)(\hat{\omega}-\underline{\omega})}\zeta_1$. The constants a_1, a_2 are equal to:

$$a_1 = \frac{\frac{1}{2}\sigma^2\eta_2(\eta_2-1)e^{(1-\eta_1)\hat{\omega}}}{(\eta_2-\eta_1)(\rho+\lambda)(\rho+\lambda-\mu-\frac{1}{2}\sigma^2)} = \frac{(\eta_2-1)e^{(1-\eta_1)\hat{\omega}}}{(\eta_2-\eta_1)(-\eta_1)(\rho+\lambda-\mu-\frac{1}{2}\sigma^2)} \tag{A24}$$

$$a_2 = -\frac{\frac{1}{2}\sigma^2\eta_1(\eta_1-1)e^{(1-\eta_2)\hat{\omega}}}{(\eta_2-\eta_1)(\rho+\lambda)(\rho+\lambda-\mu-\frac{1}{2}\sigma^2)} = \frac{(\eta_1-1)e^{(1-\eta_2)\hat{\omega}}}{(\eta_2-\eta_1)\eta_2(\rho+\lambda-\mu-\frac{1}{2}\sigma^2)} \tag{A25}$$

Once again, (A14) is the general solution to (47). Continuous differentiability of the value function at $\omega = \hat{\omega}$ yields equations (A17) and (A18). Continuous differentiability of the value function at the locus $s = 1 - e^{\theta(\omega-\hat{\omega})}$ yields equations (A19) and (A20). Then equations (48) and (49) yield:

$$\eta_1\underline{c}_1(s)e^{\eta_1(\underline{\omega}-\hat{\omega})} + \eta_2\underline{c}_2(s)e^{\eta_2(\underline{\omega}-\hat{\omega})} = (1-s)\theta\left[\underline{c}_1'(s)e^{\eta_1(\underline{\omega}-\hat{\omega})} + \underline{c}_2'(s)e^{\eta_2(\underline{\omega}-\hat{\omega})}\right]$$
$$\eta_1\hat{c}_1(s)e^{\eta_1(\underline{\omega}-\hat{\omega})} + \eta_2\hat{c}_2(s)e^{\eta_2(\underline{\omega}-\hat{\omega})} = (1-s)\theta\left[\hat{c}_1'(s)e^{\eta_1(\underline{\omega}-\hat{\omega})} + \hat{c}_2'(s)e^{\eta_2(\underline{\omega}-\hat{\omega})}\right]$$

Using the expression previously established for $\underline{c}_i$ leads to a pair of differential equations:

$$\begin{pmatrix} \hat{c}_1'(s) \\ \hat{c}_2'(s) \end{pmatrix} = \frac{1}{\theta(1-s)}\begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix}\begin{pmatrix} \hat{c}_1(s) \\ \hat{c}_2(s) \end{pmatrix} + \frac{\frac{e^{\bar{\omega}}}{\rho+\lambda-\mu-\frac{1}{2}\sigma^2} + \eta_1 e^{\eta_1\bar{\omega}}a_1 + \eta_2 e^{\eta_2\bar{\omega}}a_2}{\theta(1-s)\left(e^{\eta_1\underline{\omega}+\eta_2\bar{\omega}} - e^{\eta_1\bar{\omega}+\eta_2\underline{\omega}}\right)}\begin{pmatrix} -e^{\eta_1\hat{\omega}+\eta_2\underline{\omega}} \\ e^{\eta_1\underline{\omega}+\eta_2\hat{\omega}} \end{pmatrix}$$

The solution to this pair of ordinary differential equations is:

$$\hat{c}_1(s) = c_1(1-s)^{-\eta_1/\theta} + \zeta_1$$
$$\hat{c}_2(s) = c_2(1-s)^{-\eta_2/\theta} + \zeta_2,$$

with c_1 and c_2 two constants of integration. The value function needs to have a finite value for $s = 1$, meaning that we need to have $c_2 = 0$. Finally, we use (50) to pin down c_1. This leads to equations (A15) and (A16). Note that equations (51) are automatically satisfied. □

A.2.4.3 Low Minimum Wage

Finally we study $\hat{\omega} < \underline{\omega}$. This is equivalent to the case when $\hat{\omega} = \underline{\omega}$, since in both situations the minimum wage never binds. Applying the analysis with a moderate minimum wage gives

$$v(\omega, s) = \frac{e^{\omega}}{\rho + \lambda - \mu - \frac{1}{2}\sigma^2} + \frac{\lambda \underline{v}}{\rho + \lambda} + c_1 e^{\eta_1(\omega - \underline{\omega})} + c_2 e^{\eta_2(\omega - \underline{\omega})} \tag{A26}$$

where $\eta_1 < 0 < 1 < \eta_2$ solve (40) and

$$c_1 = \frac{-1/\eta_1}{\rho + \lambda - \mu - \frac{1}{2}\sigma^2} \left(\frac{e^{\eta_2 \bar{\omega} + (\eta_1 + 1)\underline{\omega}} - e^{(\eta_2 + \eta_1)\underline{\omega} + \bar{\omega}}}{e^{\eta_1 \underline{\omega} + \eta_2 \bar{\omega}} - e^{\eta_1 \bar{\omega} + \eta_2 \underline{\omega}}} \right) > 0 \tag{A27}$$

$$c_2 = \frac{1/\eta_2}{\rho + \lambda - \mu - \frac{1}{2}\sigma^2} \left(\frac{e^{\eta_1 \bar{\omega} + (\eta_2 + 1)\underline{\omega}} - e^{(\eta_1 + \eta_2)\underline{\omega} + \bar{\omega}}}{e^{\eta_1 \underline{\omega} + \eta_2 \bar{\omega}} - e^{\eta_1 \bar{\omega} + \eta_2 \underline{\omega}}} \right) < 0 \tag{A28}$$

Note that the value function does not depend on the worker's seniority. □

A.2.5 Monotonicity of the Value Function in s

A.2.5.1 High Minimum Wage

This is the case where $\hat{\omega} \geq \bar{\omega}$. In the region $1 > s \geq 1 - e^{\theta(\omega - \hat{\omega})}$, the partial derivative of the value function w.r.t. s takes the following form:

$$v_s(\omega, s) = \frac{1}{\theta(1-s)} \left(\frac{-\eta_1 \eta_2}{\eta_2 - \eta_1} \right) \left(\frac{e^{\hat{\omega}} - b_r}{\rho + \lambda} \right) \left(1 - e^{-(\eta_2 - \eta_1)(\hat{\omega} - \underline{\omega})} \right) (1-s)^{-\eta_1/\theta} e^{\eta_1(\omega - \hat{\omega})} > 0$$

The inequality stems from the fact that $\eta_1 < 0 < \eta_2$ and from our assumption that $e^{\hat{\omega}} > b_r$. When $0 \leq s < 1 - e^{\theta(\omega - \hat{\omega})}$, some algebra leads to:

$$v_s(\omega, s) = \frac{1}{\theta(1-s)} \left(\frac{-\eta_1 \eta_2}{\eta_2 - \eta_1} \right) \left(\frac{e^{\hat{\omega}} - b_r}{\rho + \lambda} \right) \left[(1-s)^{-\eta_2/\theta} e^{\eta_2(\omega - \hat{\omega})} - (1-s)^{-\eta_1/\theta} e^{-(\eta_2 - \eta_1)(\hat{\omega} - \underline{\omega})} e^{\eta_1(\omega - \hat{\omega})} \right]$$

First, note that $v_s(\omega, 0) > 0$ for $\omega > \underline{\omega}$, since $\eta_2(\omega - \hat{\omega}) > -(\eta_2 - \eta_1)(\hat{\omega} - \underline{\omega}) + \eta_1(\omega - \hat{\omega})$ on this interval and since $\eta_1 < 0 < \eta_2$. Similarly, evaluated at the boundary $s = 1 - e^{\theta(\omega - \hat{\omega})}$, v_s is also positive. The locus of points where $v_s(\omega, s) = 0$ is the locus of points $s = 1 - e^{\theta(\omega - \underline{\omega})}$. In the (ω, s) space, this locus of points is downward sloping, and is always below the locus of points $s = 1 - e^{\theta(\omega - \hat{\omega})}$, meaning that we have just established that v is also strictly increasing in s whenever $0 \leq s < 1 - e^{\theta(\omega - \hat{\omega})}$. □

A.2.5.2 Moderate Minimum Wage

This is the case where $\bar{\omega} > \hat{\omega} \geq \underline{\omega}$. Whether (ω, s) is in the domain $\{(\omega, s) : \bar{\omega} \geq \omega \geq \hat{\omega}, s \in [0,1)\}$ or in the domain $\{(\omega, s) : \underline{\omega} \leq \omega \leq \hat{\omega}, 1 > s \geq 1 - e^{\theta(\omega - \hat{\omega})}\}$, Appendix A.2.4 enables us to compute the partial derivative of v with respect to s:

$$v_s(\omega, s) = \frac{1}{\theta(1-s)} \left(\frac{-\eta_1\eta_2}{\eta_2 - \eta_1}\right) \left(\frac{e^{\hat{\omega}} - b_r}{\rho + \lambda}\right) (1-s)^{-\eta_1/\theta} \left(1 - e^{-(\eta_2 - \eta_1)(\hat{\omega} - \underline{\omega})}\right) e^{\eta_1(\omega - \hat{\omega})}$$

Thus on those domains, $v_s > 0$. On the interval $\underline{\omega} \leq \omega \leq \hat{\omega}$ and $s \leq 1 - e^{\theta(\omega - \hat{\omega})}$, we compute:

$$v_s(\omega, s) = \frac{1}{\theta(1-s)} \left(\frac{-\eta_1\eta_2}{\eta_2 - \eta_1}\right) \left(\frac{e^{\hat{\omega}} - b_r}{\rho + \lambda}\right) (1-s)^{-\eta_1/\theta} e^{\eta_1(\omega - \hat{\omega})} \Big[(1-s)^{-\frac{1}{\theta}(\eta_2 - \eta_1)} e^{-(\eta_2 - \eta_1)(\hat{\omega} - \omega)} - e^{-(\eta_2 - \eta_1)(\hat{\omega} - \underline{\omega})}\Big]$$

$v_s(\omega, 0) > 0$ when $\underline{\omega} < \omega \leq \hat{\omega}$, since $-(\eta_2 - \eta_1)(\hat{\omega} - \omega) > -(\eta_2 - \eta_1)(\hat{\omega} - \underline{\omega})$ on this interval. Similarly, evaluated at the boundary $s = 1 - e^{\theta(\omega - \hat{\omega})}$, v_s is also positive. The locus of points where $v_s(\omega, s) = 0$ is the locus of points $s = 1 - e^{\theta(\omega - \underline{\omega})}$. In the (ω, s) space, this locus of points is downward sloping, and is always below the locus of points $s = 1 - e^{\theta(\omega - \hat{\omega})}$, meaning that we have just established that v is also strictly increasing in s whenever $\underline{\omega} \leq \omega \leq \hat{\omega}$ and $0 < s \leq 1 - e^{\theta(\omega - \hat{\omega})}$. □

A.2.5.3 Low Minimum Wage

The closed form expression for v established in Appendix A.2.4 is independent of s, which proves the claim. □

A.2.6 Monotonicity of the Value Function in ω

A.2.6.1 High or Moderate Minimum Wage

A high or moderate minimum wage corresponds to $\hat{\omega} \geq \underline{\omega}$. In this case, we established in A.2.5 that the function v is strictly increasing in s, for all values $\omega \in [\underline{\omega}, \bar{\omega}]$. Let us take an arbitrary seniority level $s \in [0, 1)$. Since $v_s(\underline{\omega}, s) > 0$ and $v_s(\bar{\omega}, s) > 0$, we can use (48) and (49) to conclude that $v_\omega(\underline{\omega}, s) > 0$ and $v_\omega(\bar{\omega}, s) > 0$. We will prove by contradiction that the value function must be strictly increasing in ω, for all $\omega \in [\underline{\omega}, \bar{\omega}]$. Let us assume for now that v is not strictly increasing in ω on that interval. Since v is twice differentiable in ω on $[\underline{\omega}, \bar{\omega}]$, and since $v_\omega(\underline{\omega}, s)$ and $v_\omega(\bar{\omega}, s)$ are both strictly positive, two situations can arise if v is not strictly increasing in ω:

1. There exists an open interval on which the function $v(\cdot, s)$ is strictly decreasing, in other words there exists (ω_1, ω_2), with $\underline{\omega} < \omega_1 < \omega_2 < \bar{\omega}$, such that $v(\omega_1, s) > v(\omega_2, s)$, with $v_\omega(\omega_1, s) = v_\omega(\omega_2, s) = 0$, and $v_{\omega\omega}(\omega_1, s) < 0 < v_{\omega\omega}(\omega_2, s)$.
2. There exists an open interval on which the function $v(\cdot, s)$ is constant, in other words there exists (ω_1, ω_2), with $\underline{\omega} < \omega_1 < \omega_2 < \bar{\omega}$, such that for all $\omega \in (\omega_1, \omega_2)$, all the partial derivatives of v with respect to ω evaluated at those points are equal to zero.

We will study these cases one at a time.

1. In this case, since $v_\omega(\omega_1, s) = v_\omega(\omega_2, s) = 0$, we can use (47) to establish the following inequality:

$$v_{\omega\omega}(\omega_1, s) = \frac{2}{\sigma^2}\left((\lambda + \rho)v(\omega_1, s) - (R(\omega_1, s) + \lambda \underline{v})\right) < 0$$
$$v_{\omega\omega}(\omega_2, s) = \frac{2}{\sigma^2}\left((\lambda + \rho)v(\omega_2, s) - (R(\omega_2, s) + \lambda \underline{v})\right) > 0$$

   Both inequalities lead to:

$$v(\omega_1, s) < \frac{1}{\lambda + \rho}\left(R(\omega_1, s) + \lambda \underline{v}\right)$$
$$v(\omega_2, s) > \frac{1}{\lambda + \rho}\left(R(\omega_2, s) + \lambda \underline{v}\right)$$

   But this is a contradiction, since we know that $R(\omega_2, s) \geq R(\omega_1, s)$ and since we have assumed $v(\omega_1, s) > v(\omega_2, s)$.

2. In thise case, we know that for any $\tilde{\omega} \in (\omega_1, \omega_2)$, we have $v_\omega(\tilde{\omega}, s) = v_{\omega\omega}(\tilde{\omega}, s) = v_{\omega\omega\omega}(\tilde{\omega}, s) = ... = 0$. Indeed, we know that the function v is constant in ω on that interval and that the function v is $\mathcal{C}^\infty$ in ω given the closed form expression established for v. This provides for an infinite number of algebraic and linearly independent equations satisfied by $\tilde{\omega}$, which leads to an immediate contradiction.

Thus we found a contradiction in both situations, and have proved our claim. Note that an alternative proof can be constructed as follows. Note $\xi(\cdot)$ the Dirac-Delta function.

For any function $f(\cdot)$, any $\epsilon > 0$ and any a, we have $\int_{a-\epsilon}^{a+\epsilon} f(x)\xi(x-a)dx = f(x)$. Then differentiate (47) w.r.t. ω to obtain the following:

$$(\rho+\lambda)v_\omega(\omega,s) = e^\omega 1_{\{\omega\geq\hat{\omega}\}} + \xi\left(s-1+e^{\theta(\omega-\hat{\omega})}\right) + \xi\left(\omega-\hat{\omega}\right) + \mu v_{\omega\omega}(\omega,s) + \tfrac{1}{2}\sigma^2 v_{\omega\omega\omega}(\omega,s)$$

It is easy to make this argument more rigorous by approximating the Dirac-Delta function by a sequence of smooth functions and let such sequence converge to the Dirac-Delta function. Using Feynman-Kac, v_ω thus admits the integral representation:

$$v_\omega(\omega,s) = \mathbb{E}_\omega\left[\int_0^\tau e^{-(\rho+\lambda)t}\left(e^{\omega(t))}1_{\{\omega(t)\geq\hat{\omega}\}} + \xi\left(s-1+e^{\theta(\omega(t)-\hat{\omega})}\right) + \xi\left(\omega(t)-\hat{\omega}\right)\right)dt \right.$$
$$\left. + e^{-(\rho+\lambda)\tau}v_\omega(\omega(\tau),s)\right]$$

In the above, $\tau = \tau(\underline{\omega},s)\wedge\tau(\bar{\omega},s)$. We established in A.2.5 that the function v is strictly increasing in s, for all values $\omega\in[\underline{\omega},\bar{\omega}]$. Thus we can use (48) and (49) to conclude that $v_\omega(\underline{\omega},s) > 0$ and $v_\omega(\bar{\omega},s) > 0$. In other words $v_\omega(\omega,s)$ can be represented as the expectation of the sum of (a) an integral of positive terms and (b) some discounted term that is positive. This means that v_ω is strictly positive. □

A.2.6.2 Low Minimum Wage

This case was treated in Alvarez and Shimer (2011) and is thus omitted. □

A.2.7 Existence of Equilibrium

A.2.7.1 High Minimum Wage

In Appendix A.2.4, we establish a closed form expression for the value function under the assumption that $\hat{\omega}\geq\bar{\omega}$. We now need to show that there exists a threshold value $\hat{\omega}^*$ such that if $\hat{\omega} > \hat{\omega}^*$, the boundary conditions (43) and (44) lead to unique values $\underline{\omega}(\hat{\omega}),\bar{\omega}(\hat{\omega})$ that are consistent with $\underline{\omega}(\hat{\omega}) < \bar{\omega}(\hat{\omega}) < \hat{\omega}$. Using Appendix A.2.4, (43) can be expressed as follows:

$$v(\underline{\omega},0;\underline{\omega},\bar{\omega},\hat{\omega}) = \frac{b_r+\lambda\underline{v}}{\rho+\lambda} + \left(\frac{e^{\hat{\omega}}-b_r}{\rho+\lambda}\right)e^{-\eta_2(\hat{\omega}-\underline{\omega})} = \underline{v} \tag{A29}$$

Similarly, (44) can be expressed as follows:

$$v(\bar{\omega},0;\underline{\omega},\bar{\omega},\hat{\omega}) = \frac{b_r+\lambda\underline{v}}{\rho+\lambda} + \left(\frac{e^{\hat{\omega}}-b_r}{\rho+\lambda}\right)\left[\frac{\eta_2}{\eta_2-\eta_1}e^{-(\eta_2-\eta_1)(\hat{\omega}-\underline{\omega})}e^{\eta_1(\bar{\omega}-\hat{\omega})} + \frac{-\eta_1}{\eta_2-\eta_1}e^{\eta_2(\bar{\omega}-\hat{\omega})}\right] = \bar{v} \quad \text{(A30)}$$

We first show that irrespective of the value of $\hat{\omega}$, the system of 2 equations (A29) and (A30) in two unknown $\underline{\omega},\bar{\omega}$ always has a unique solution. Let $\bar{\omega}^*$ be the point that solves $v(\bar{\omega}^*,0;\bar{\omega}^*,\bar{\omega}^*,\hat{\omega}) = \bar{v}$. In other words, $\bar{\omega}^*$ solves[9]:

$$\frac{b_r+\lambda\underline{v}}{\rho+\lambda} + \frac{e^{\hat{\omega}}-b_r}{\rho+\lambda}e^{-\eta_2(\hat{\omega}-\bar{\omega}^*)} = \bar{v} \quad \text{(A31)}$$

Now take $\underline{\omega} < \bar{\omega}^*$. The function $\bar{\omega} \to v(\bar{\omega},0;\underline{\omega},\bar{\omega},\hat{\omega})$ is strictly increasing in $\bar{\omega}$, and converges to $+\infty$ when $\bar{\omega} \to +\infty$. Additionally, since $\underline{\omega} \to v(\bar{\omega},0;\underline{\omega},\bar{\omega},\hat{\omega})$ is strictly increasing in $\underline{\omega}$, it must be the case that $v(\bar{\omega}^*,0;\underline{\omega},\bar{\omega}^*,\hat{\omega}) < \bar{v}$, since $\underline{\omega} < \bar{\omega}^*$. Thus, the intermediate value theorem delivers us $\bar{\Omega}(\underline{\omega}) > \bar{\omega}^*$, which verifies $v(\bar{\Omega}(\underline{\omega}),0;\underline{\omega},\bar{\Omega}(\underline{\omega}),\hat{\omega}) = \bar{v}$. An example of function $\bar{\Omega}(\cdot)$ is plotted in Figure A-1. Continuity of v gives us continuity of $\bar{\Omega}(\cdot)$, and monotonicity of v tells us that $\bar{\Omega}(\cdot)$ is strictly decreasing. Moreover, $\lim_{\underline{\omega}\to-\infty}\bar{\Omega}(\underline{\omega})$ exists and is equal to $\bar{\omega}^{**}$ (with $\bar{\omega}^{**} > \bar{\omega}^*$), which solves[10]:

$$\frac{b_r+\lambda\underline{v}}{\rho+\lambda} + \left(\frac{e^{\hat{\omega}}-b_r}{\rho+\lambda}\right)\left(\frac{-\eta_1}{\eta_2-\eta_1}\right)e^{-\eta_2(\hat{\omega}-\bar{\omega}^{**})} = \bar{v} \quad \text{(A32)}$$

Notice that $\bar{\Omega}(\underline{\omega}) \in (\bar{\omega}^*,\bar{\omega}^{**})$ for any $\underline{\omega} < \bar{\omega}^*$. Let $\underline{\omega}^*$ the point that solves $v(\underline{\omega}^*,0;\underline{\omega}^*,\underline{\omega}^*,\hat{\omega}) = \underline{v}$. In other words, $\underline{\omega}^*$ solves:

$$\frac{b_r+\lambda\underline{v}}{\rho+\lambda} + \frac{e^{\hat{\omega}}-b_r}{\rho+\lambda}e^{-\eta_2(\hat{\omega}-\underline{\omega}^*)} = \underline{v} \quad \text{(A33)}$$

Note that we must have $\underline{\omega}^* < \bar{\omega}^*$, since $\underline{v} < \bar{v}$. Note also that for any $\bar{\omega} > \underline{\omega}^*$, $v(\underline{\omega}^*,0;\bar{\omega},\underline{\omega}^*,\hat{\omega}) = \underline{v}$. In other words, the implicit function $\underline{\Omega}(\cdot)$ which verifies $v(\underline{\Omega}(\bar{\omega}),0;\bar{\omega},\underline{\Omega}(\bar{\omega}),\hat{\omega})$ $\underline{v}$ for all $\bar{\omega} > \underline{\omega}^*$ is a constant function, equal to $\underline{\omega}^*$. Note finally that we have the fol-

[9]One can verify that $\bar{\omega}^* = \hat{\omega} + \frac{1}{\eta_2}\ln\left[\left(\frac{b_i-b_r}{e^{\hat{\omega}}-b_r}\right) + \frac{\rho+\lambda}{\alpha}\left(\frac{b_i-b_s}{e^{\hat{\omega}}-b_r}\right)\right]$

[10]One can verify that $\bar{\omega}^{**} = \bar{\omega}^* + \frac{1}{\eta_2}\ln\left(\frac{\eta_2-\eta_1}{-\eta_1}\right)$

lowing analytical expression:

$$\underline{\omega}^* = \hat{\omega} + \frac{1}{\eta_2} \log\left(\frac{b_i - b_r}{e^{\hat{\omega}} - b_r}\right) \tag{A34}$$

In other words, it must be the case that $b_i > b_r$ and $b_i < e^{\hat{\omega}}$ for an equilibrium of this type to exist. An example of function $\underline{\Omega}(\cdot)$ is plotted in Figure A-1.

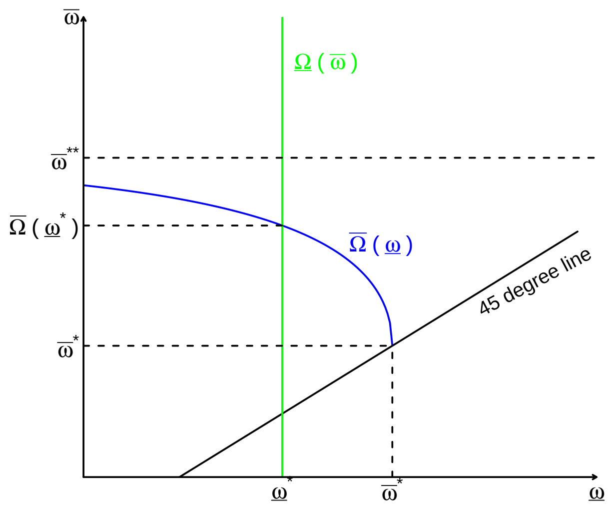

Figure A-1: **Equilibrium with high minimum wage.**

An equilibrium is a fixed point of the composition $\bar{\Omega} \circ \underline{\Omega}$. Since $\underline{\Omega}(\cdot)$ is a constant function, the composition $\bar{\Omega} \circ \underline{\Omega}$ is also constant, and there is a unique fixed point $\bar{\omega}$ of the function $\bar{\Omega} \circ \underline{\Omega}$, attained for $\bar{\omega} = \bar{\Omega}(\underline{\omega}^*) \in (\bar{\omega}^*, \bar{\omega}^{**})$.

We have thus proven that irrespective of $\hat{\omega}$, the system of 2 equations (43) and (44) in two unknown $\underline{\omega}, \bar{\omega}$ always has a unique solution. It remains to check that the cutoff $\bar{\omega} = \bar{\Omega}(\underline{\omega}^*)$ is indeed such that $\bar{\omega} < \hat{\omega}$, or at least establish a sufficient condition such that it is the case. Note that $\bar{\omega}$ verifies the equation:

$$v(\bar{\omega}, 0; \underline{\omega}^*, \bar{\omega}, \hat{\omega}) = \frac{b_r + \lambda \underline{v}}{\rho + \lambda} + \left(\frac{e^{\hat{\omega}} - b_r}{\rho + \lambda}\right) \left[\frac{\eta_2}{\eta_2 - \eta_1} e^{-(\eta_2 - \eta_1)(\hat{\omega} - \underline{\omega}^*)} e^{\eta_1(\bar{\omega} - \hat{\omega})} + \frac{-\eta_1}{\eta_2 - \eta_1} e^{\eta_2(\bar{\omega} - \hat{\omega})}\right] = \bar{v}$$

Where $\underline{\omega}^*$ has been analytically determined in (A34). Reinjecting into the equation

$v(\bar{\omega}, 0; \underline{\omega}^*, \bar{\omega}, \hat{\omega}) = \bar{v}$ and simplifying, we obtain:

$$\frac{\eta_2}{\eta_2 - \eta_1} \left(\frac{e^{\hat{\omega}} - b_r}{b_i - b_r} \right)^{\frac{\eta_1}{\eta_2}} e^{\eta_1(\bar{\omega} - \hat{\omega})} + \frac{-\eta_1}{\eta_2 - \eta_1} \left(\frac{e^{\hat{\omega}} - b_r}{b_i - b_r} \right) e^{\eta_2(\bar{\omega} - \hat{\omega})} = 1 + \frac{\rho + \lambda}{\alpha} \frac{b_i - b_s}{b_i - b_r} \quad \text{(A35)}$$

If there is no solution $\bar{\omega} < \hat{\omega}$ of the equation above, this leads to a contradiction, establishing that if an equilibrium exists, it must be such that $\bar{\omega} > \hat{\omega}$. Note that (A35) can be rewritten $\phi(\bar{\omega}, \hat{\omega}) = 1 + \frac{\rho+\lambda}{\alpha} \frac{b_i - b_s}{b_i - b_r}$, where we have defined

$$\phi(\bar{\omega}, \hat{\omega}) \equiv \frac{\eta_2}{\eta_2 - \eta_1} \left(\frac{e^{\hat{\omega}} - b_r}{b_i - b_r} \right)^{\frac{\eta_1}{\eta_2}} e^{\eta_1(\bar{\omega} - \hat{\omega})} + \frac{-\eta_1}{\eta_2 - \eta_1} \left(\frac{e^{\hat{\omega}} - b_r}{b_i - b_r} \right) e^{\eta_2(\bar{\omega} - \hat{\omega})}$$

The function ϕ is convex in $\bar{\omega}$, and:

$$\phi_{\bar{\omega}}(\bar{\omega}, \hat{\omega}) = \left(\frac{e^{\hat{\omega}} - b_r}{b_i - b_r} \right) \left(\frac{-\eta_1 \eta_2}{\eta_2 - \eta_1} \right) \left[e^{\eta_2(\bar{\omega} - \hat{\omega})} - \left(\frac{e^{\hat{\omega}} - b_r}{b_i - b_r} \right)^{-\frac{\eta_2 - \eta_1}{\eta_2}} e^{\eta_1(\bar{\omega} - \hat{\omega})} \right]$$

Thus $\phi(\cdot, \hat{\omega})$ reaches its minimum (in $\bar{\omega}$) for $\underline{\omega}^*(\hat{\omega})$ (see (A34)), and the minimum reached is equal to:

$$\phi(\underline{\omega}^*(\hat{\omega}), \hat{\omega}) = 1 < 1 + \frac{\rho + \lambda}{\alpha} \frac{b_i - b_s}{b_i - b_r}$$

Thus, (A35), for the unknown $\bar{\omega}$, always has two solutions. But since the minimum of this function is reached at $\underline{\omega}^*(\hat{\omega})$, only the larger root of this equation can be a candidate $\bar{\omega}$. Let $\bar{\omega}(\hat{\omega})$ be the largest of these two roots (and the only legitimate candidate equilibrium). Note that the implicit function theorem and some algebra leads us to calculate:

$$\bar{\omega}'(\hat{\omega}) = 1 - \frac{1}{\eta_2} \frac{e^{\hat{\omega}}}{e^{\hat{\omega}} - b_r} \quad \text{(A36)}$$

The functions $\bar{\omega}(\cdot)$ is thus convex. Let $\hat{\omega}_0$ solve $\bar{\omega}'(\hat{\omega}_0) = 0$, in other words $e^{\hat{\omega}_0} = \frac{\eta_2}{\eta_2 - 1} b_r$. We thus know that $\bar{\omega}(\cdot)$ is decreasing for $\hat{\omega} < \hat{\omega}_0$ and is increasing for $\hat{\omega} > \hat{\omega}_0$. Notice also that $\lim_{\hat{\omega} \searrow \log b_r} \bar{\omega}(\hat{\omega}) = +\infty$ and that $\lim_{\hat{\omega} \to +\infty} \bar{\omega}'(\hat{\omega}) = 1 - 1/\eta_2 < 1$. Let $\hat{\omega}^*$ be the unique fixed point of $\bar{\omega}(\cdot)$, in other words $\hat{\omega}^* = \bar{\omega}(\hat{\omega}^*)$. In Figure A-2, we provide an illustration of what $\bar{\omega}(\cdot)$ and $\underline{\omega}(\cdot)$ look like. We have thus proven that the threshold value $\hat{\omega}^*$ is such that:

1. If $\hat{\omega} > \hat{\omega}^*$, the highest solution to (A35) satisfies $\hat{\omega} > \bar{\omega}(\hat{\omega})$, meaning there is a unique equilibrium of the form above in such labor market;
2. If $\hat{\omega} < \hat{\omega}^*$, the highest solution to (A35) satisfies $\bar{\omega}(\hat{\omega}) > \hat{\omega}$, meaning there is no equilibrium of the form specified above in such labor market.

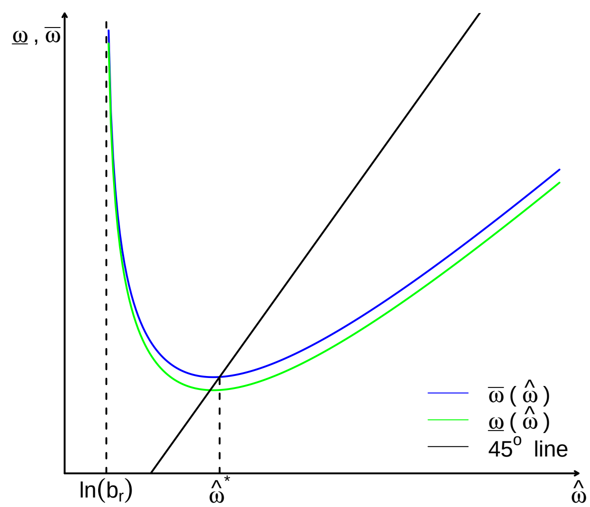

Figure A-2: **Endogeneous equilibrium boundaries with high minimum wages.**

Note that $\hat{\omega}^*$ satisfies $\phi\left(\hat{\omega}^*, \hat{\omega}^*\right) = 1 + \frac{\rho+\lambda}{\alpha}\frac{b_i - b_s}{b_i - b_r}$, in other words $\hat{\omega}^*$ satisfies:

$$\frac{\eta_2}{\eta_2-\eta_1}\left(\frac{e^{\hat{\omega}^*}-b_r}{b_i-b_r}\right)^{\frac{\eta_1}{\eta_2}} + \frac{-\eta_1}{\eta_2-\eta_1}\left(\frac{e^{\hat{\omega}^*}-b_r}{b_i-b_r}\right) - \left(1+\frac{\rho+\lambda}{\alpha}\frac{b_i-b_s}{b_i-b_r}\right) = 0 \qquad \text{(A37)}$$

The function of $\hat{\omega}^*$ defined via (A37) is convex, and reaches its minimum $-\frac{\rho+\lambda}{\alpha}\frac{b_i-b_s}{b_i-b_r} < 0$ at $\hat{\omega}^* = \ln b_i$. Thus this equation has two roots. Now, remember that for an equilibrium of this type to exist, it must be the case that $e^{\hat{\omega}} > b_i$. In other words, the root that we are looking for is the highest of the two roots of the equation above. Using this notation, we can also use the differential equation (A36) satisfied by $\bar{\omega}(\cdot)$ to obtain the following closed-form expressions:

$$\bar{\omega}(\hat{\omega}) = \hat{\omega} - \frac{1}{\eta_2}\ln\left(\frac{e^{\hat{\omega}}-b_r}{e^{\hat{\omega}^*}-b_r}\right)$$

$$\underline{\omega}(\hat{\omega}) = \hat{\omega} - \frac{1}{\eta_2} \ln \left(\frac{e^{\hat{\omega}} - b_r}{b_i - b_r} \right)$$

□

A.2.7.2 Moderate Minimum Wage

We establish the existence of an equilibrium with moderate minimum wage whenever $\omega \in ({}_*\hat{\omega}, \hat{\omega}^*)$ via several lemmas.

LEMMA 2. The value function $v(\omega, s; \underline{\omega}, \bar{\omega}, \hat{\omega})$ is increasing in $\bar{\omega}$.

The proof of this lemma is straightforward. Inspection of (A14) shows that the value function admits a partial derivative with respect to $\bar{\omega}$ that is equal to:

$$v_{\bar{\omega}}(\omega, s; \underline{\omega}, \bar{\omega}, \hat{\omega}) = \frac{\partial \zeta_1}{\partial \bar{\omega}} \left(1 - \frac{\eta_1}{\eta_2} e^{(\eta_2 - \eta_1)(\omega - \underline{\omega})} \right) e^{\eta_1(\omega - \hat{\omega})},$$

for any $\omega \in [\underline{\omega}, \bar{\omega}]$ and any $s \in [0, 1]$. Some algebra enables us to express ζ_1 as follows:

$$\zeta_1 = \frac{e^{\eta_1 \hat{\omega}}}{(-\eta_1)(\eta_2 - \eta_1)\left(\rho + \lambda - \mu - \frac{1}{2}\sigma^2\right)} A(\underline{\omega}, \bar{\omega}, \hat{\omega}) \tag{A38}$$

Where we have defined:

$$A(\underline{\omega}, \bar{\omega}, \hat{\omega}) \equiv \frac{e^{\eta_2 \underline{\omega}} \left[\left((\eta_2 - 1)e^{\eta_1(\bar{\omega} - \hat{\omega})} + (1 - \eta_1)e^{\eta_2(\bar{\omega} - \hat{\omega})} \right) e^{\hat{\omega}} - (\eta_2 - \eta_1)e^{\bar{\omega}} \right]}{e^{\eta_1 \underline{\omega} + \eta_2 \bar{\omega}} - e^{\eta_1 \bar{\omega} + \eta_2 \underline{\omega}}} \tag{A39}$$

Notice that A is always positive: its denominator is positive, and the terms in bracket can be re-written, for $x = \bar{\omega} - \hat{\omega} \geq 0$:

$$\begin{aligned} &\left((\eta_2 - 1)e^{\eta_1(\bar{\omega} - \hat{\omega})} + (1 - \eta_1)e^{\eta_2(\bar{\omega} - \hat{\omega})} \right) e^{\hat{\omega}} - (\eta_2 - \eta_1)e^{\bar{\omega}} = \\ &\qquad e^{\hat{\omega} + x} \left[(\eta_2 - 1)e^{(\eta_1 - 1)x} + (1 - \eta_1)e^{(\eta_2 - 1)x} - (\eta_2 - \eta_1) \right] \end{aligned}$$

The term in brackets is positive since it is an increasing function of x, with value 0 at

$x = 0$. It can be showed that:

$$A_{\bar{\omega}}(\underline{\omega}, \bar{\omega}, \hat{\omega}) = \frac{e^{\eta_2\underline{\omega}+\bar{\omega}}\,(\eta_2 - \eta_1)\left[(\eta_2 - 1)(1 - e^{(\eta_1-1)(\bar{\omega}-\hat{\omega})})e^{\eta_1\underline{\omega}+\eta_2\bar{\omega}} + (1 - \eta_1)(1 - e^{(\eta_2-1)(\bar{\omega}-\hat{\omega})})e^{\eta_1\bar{\omega}+\eta}\right.}{(e^{\eta_1\underline{\omega}+\eta_2\bar{\omega}} - e^{\eta_1\bar{\omega}+\eta_2\underline{\omega}})^2}$$

And since the function $g(x) = (\eta_2 - 1)(1 - e^{(\eta_1-1)x})e^{\eta_1\underline{\omega}+\eta_2\bar{\omega}} + (1 - \eta_1)(1 - e^{(\eta_2-1)x})e^{\eta_1\bar{\omega}+\eta_2\underline{\omega}}$ is such that $g(0) = 0$ and for $x \in [0, \bar{\omega} - \underline{\omega}]$:

$$g'(x) = (\eta_2 - 1)(1 - \eta_1)\left(e^{\eta_1\underline{\omega}+\eta_2\bar{\omega}+(\eta_1-1)x} - e^{\eta_1\bar{\omega}+\eta_2\underline{\omega}+(\eta_2-1)x}\right) \geq 0$$

It must be the case that $g(x) \geq 0$ for all $x \in [0, \bar{\omega} - \underline{\omega}]$, meaning that we must have $A_{\bar{\omega}}(\underline{\omega}, \bar{\omega}, \hat{\omega}) \geq 0$. This means that $v_{\bar{\omega}} \geq 0$. $\square$

LEMMA 3. The value function $v(\omega, s; \underline{\omega}, \bar{\omega}, \hat{\omega})$ is increasing in $\underline{\omega}$.

Our proof is similar to the proof of the previous lemma. Inspection of (A14) shows that the value function admits a partial derivative with respect to $\underline{\omega}$ that is equal to:

$$v_{\underline{\omega}}(\omega, s; \underline{\omega}, \bar{\omega}, \hat{\omega}) = \frac{d}{d\underline{\omega}}\left[\left(1 - \frac{\eta_1}{\eta_2}e^{(\eta_2-\eta_1)(\omega-\underline{\omega})}\right)\zeta_1 + c_1(1-s)^{-\eta_1/\theta}\right]e^{\eta_1(\omega-\hat{\omega})},$$

for any $\omega \in [\underline{\omega}, \bar{\omega}]$ and any $s \in [0, 1]$. In other words, we have:

$$v_{\underline{\omega}}(\omega, s) = \left[\frac{\partial\zeta_1}{\partial\underline{\omega}}\left(1 - \frac{\eta_1}{\eta_2}e^{(\eta_2-\eta_1)(\omega-\underline{\omega})}\right) + \frac{\eta_1}{\eta_2}(\eta_2 - \eta_1)e^{(\eta_2-\eta_1)(\omega-\underline{\omega})}\zeta_1 \right.$$
$$\left. + \,\eta_2\frac{e^{\hat{\omega}} - b_r}{\rho + \lambda}e^{-(\eta_2-\eta_1)(\hat{\omega}-\underline{\omega})}(1-s)^{-\eta_1/\theta}\right]e^{\eta_1(\omega-\hat{\omega})}$$

The last term of the expression in brackets is clearly positive. Remember that we expressed ζ_1 as a constant (independent of $\underline{\omega}$) multiplied by the function A, defined in (A39). Note also that:

$$A_{\underline{\omega}}(\underline{\omega}, \bar{\omega}, \hat{\omega}) = \frac{(\eta_2 - \eta_1)e^{\eta_1\underline{\omega}+\eta_2\bar{\omega}}A(\underline{\omega}, \bar{\omega}, \hat{\omega})}{e^{\eta_1\underline{\omega}+\eta_2\bar{\omega}} - e^{\eta_1\bar{\omega}+\eta_2\underline{\omega}}}$$

This enables us to simplify the first two terms in the expression of $v_{\underline{\omega}}$ (up to a constant factor):

$$\left(1-\frac{\eta_1}{\eta_2}e^{(\eta_2-\eta_1)(\omega-\underline{\omega})}\right)A_{\underline{\omega}}(\underline{\omega},\bar{\omega},\hat{\omega})+\frac{\eta_1}{\eta_2}(\eta_2-\eta_1)e^{(\eta_2-\eta_1)(\omega-\underline{\omega})}A(\underline{\omega},\bar{\omega},\hat{\omega})=$$
$$\frac{(\eta_2-\eta_1)e^{\eta_1\underline{\omega}+\eta_2\bar{\omega}}}{e^{\eta_1\underline{\omega}+\eta_2\bar{\omega}}-e^{\eta_1\bar{\omega}+\eta_2\underline{\omega}}}\left(1-\frac{\eta_1}{\eta_2}e^{-(\eta_2-\eta_1)(\bar{\omega}-\omega)}\right)A(\hat{\omega},\underline{\omega},\bar{\omega})$$

This latter expression is clearly positive, establishing the claim. □

LEMMA 4. Assume that $\hat{\omega}>\ln\left(\frac{\eta_2-1}{\eta_2}b_i\right)$. Let $a(\hat{\omega})$ be defined as follows:

$$a(\hat{\omega})=\begin{cases}\hat{\omega} & \text{if } \hat{\omega}\geq\ln b_i\\ a(\hat{\omega})>\hat{\omega}:v\left(\hat{\omega},0;\hat{\omega},a(\hat{\omega}),\hat{\omega}\right)=\underline{v} & \text{if } \ln\left(\frac{\eta_2-1}{\eta_2}b_i\right)<\hat{\omega}<\ln b_i\end{cases}$$

Then for any $\bar{\omega}>a(\hat{\omega})$, we have $v\left(\hat{\omega},0;\hat{\omega},\bar{\omega},\hat{\omega}\right)>\underline{v}$. There exists a unique $\underline{\omega}=\underline{\Gamma}\left(\bar{\omega}\right)\leq\hat{\omega}$ that satisfies $v\left(\underline{\omega},0;\underline{\omega},\bar{\omega},\hat{\omega}\right)=\underline{v}$. The function $\underline{\Gamma}$ has domain $[a(\hat{\omega});+\infty)$. $\underline{\Gamma}$ is decreasing in $\bar{\omega}$, with limits:

$$\lim_{\bar{\omega}\to a(\hat{\omega})}\underline{\Gamma}\left(\bar{\omega}\right)=\min\left(\hat{\omega},\hat{\omega}+\frac{1}{\eta_2}\ln\left(\frac{b_i-b_r}{e^{\hat{\omega}}-b_r}\right)\right)$$
$$\lim_{\bar{\omega}\to+\infty}\underline{\Gamma}\left(\bar{\omega}\right)=\hat{\omega}+\frac{1}{\eta_2}\ln\left(\frac{b_i-b_r}{\frac{\eta_2}{\eta_2-1}e^{\hat{\omega}}-b_r}\right)$$

Use the formula for v in (A14) and $\underline{\omega}=\hat{\omega}$ to notice that:

$$v\left(\hat{\omega},0;\hat{\omega},\bar{\omega},\hat{\omega}\right)=\frac{1}{\rho+\lambda}\left[e^{\hat{\omega}}+\lambda\underline{v}+\frac{e^{\eta_1\hat{\omega}}A\left(\hat{\omega},\bar{\omega},\hat{\omega}\right)}{(1-\eta_1)(\eta_2-1)}\right]$$

In the above, $A\left(\underline{\omega},\bar{\omega},\hat{\omega}\right)$ is defined via (A39), and it is always positive when $\bar{\omega}\geq\hat{\omega}$. Thus, if $e^{\hat{\omega}}\geq b_i$, then $v\left(\hat{\omega},0;\hat{\omega},\bar{\omega},\hat{\omega}\right)\geq\underline{v}$. If $e^{\hat{\omega}}<b_i$, we have $v\left(\hat{\omega},0;\hat{\omega},\hat{\omega},\hat{\omega}\right)<\underline{v}$. However, notice that $\bar{\omega}\to A\left(\hat{\omega},\bar{\omega},\hat{\omega}\right)$ is increasing, and that:

$$\lim_{\bar{\omega}\to+\infty}v\left(\hat{\omega},0;\hat{\omega},\bar{\omega},\hat{\omega}\right)=\frac{1}{\rho+\lambda}\left[\frac{\eta_2}{\eta_2-1}e^{\hat{\omega}}+\lambda\bar{v}\right]$$

Thus, so long as $e^{\hat{\omega}}>\frac{\eta_2-1}{\eta_2}b_i$, it is possible to pick $\bar{\omega}$ high enough such that $v\left(\hat{\omega},0;\hat{\omega},\bar{\omega},\hat{\omega}\right)\geq\underline{v}$. Let $a(\hat{\omega})>\hat{\omega}$ be such that $v\left(\hat{\omega},0;\hat{\omega},a(\hat{\omega}),\hat{\omega}\right)=\underline{v}$: $a(\hat{\omega})$ will be the lowest bound of the domain of the function $\underline{\Gamma}$ to be described shortly, and is defined implicitly via:

$$(1-\eta_1)(\eta_2-1)\left(e^{\eta_2(a(\hat{\omega})-\hat{\omega})}-e^{\eta_1(a(\hat{\omega})-\hat{\omega})}\right)b_i = \\ \left[\eta_2(1-\eta_1)e^{\eta_2(a(\hat{\omega})-\hat{\omega})}+\eta_1(\eta_2-1)e^{\eta_1(a(\hat{\omega})-\hat{\omega})}-(\eta_2-\eta_1)e^{a(\hat{\omega})-\hat{\omega}}\right]e^{\hat{\omega}}$$

$a(\cdot)$ is decreasing for $\hat{\omega}\in\left(\ln\left(\frac{\eta_2-1}{\eta_2}b_i\right);\ln b_i\right)$, with the following limits:

$$\lim_{\hat{\omega}\to\ln\left(\frac{\eta_2-1}{\eta_2}b_i\right)} a(\hat{\omega}) = +\infty$$
$$\lim_{\hat{\omega}\to\ln b_i} a(\hat{\omega}) = \ln b_i$$

Notice also that $a(\cdot)$ is none other than the implicit function that solves:

$$v\left(\underline{\omega};\underline{\omega},a(\underline{\omega})\right)=\underline{v},$$

where v is the value function computed in the case of a low minimum wage (i.e. never binding) displayed in (A26).

Then, notice that $\lim_{\underline{\omega}\to-\infty} v\left(\underline{\omega},0;\underline{\omega},\bar{\omega},\hat{\omega}\right)=\frac{b_r+\lambda\underline{v}}{\rho+\lambda}<\underline{v}$ since $\underline{v}=b_i/\rho$ and since $b_r<b_i$ by assumption. Since v is increasing in ω and in $\underline{\omega}$, it means that the function $\underline{\omega}\to v\left(\underline{\omega},0;\underline{\omega},\bar{\omega},\hat{\omega}\right)$ is strictly increasing, in addition to being continuous. Thus, the intermediate value theorem provides for the existence and uniqueness of $\underline{\omega}=\underline{\Gamma}\left(\bar{\omega}\right)\leq\hat{\omega}$ such that $v\left(\underline{\omega},0;\underline{\omega},\bar{\omega},\hat{\omega}\right)=\underline{v}$. Since v is increasing in $\bar{\omega}$, the function $\underline{\Gamma}$ must be decreasing in $\bar{\omega}$. Notice then that the equation $v\left(\underline{\omega},0\right)=\underline{v}$ can be expressed as follows:

$$\begin{aligned}\underline{v}&=\frac{b_r+\lambda\underline{v}}{\rho+\lambda}+\left(\frac{e^{\hat{\omega}}-b_r}{\rho+\lambda}\right)e^{-\eta_2(\hat{\omega}-\underline{\omega})}+\zeta_1\left(1-\frac{\eta_1}{\eta_2}\right)e^{-\eta_1(\hat{\omega}-\underline{\omega})}\\&=\frac{b_r+\lambda\underline{v}}{\rho+\lambda}+\left(\frac{e^{\hat{\omega}}-b_r}{\rho+\lambda}\right)e^{-\eta_2(\hat{\omega}-\underline{\omega})}+\frac{e^{\eta_1\underline{\omega}}A\left(\underline{\omega},\bar{\omega},\hat{\omega}\right)}{(\rho+\lambda)\left(1-\eta_1\right)\left(\eta_2-1\right)}\end{aligned}$$

If $\hat{\omega}\geq\ln b_i$, when $\bar{\omega}\to\hat{\omega}$, since $A\left(\underline{\omega},\bar{\omega},\hat{\omega}\right)\to 0$, we have:

$$\lim_{\bar{\omega}\to\hat{\omega}}\underline{\Gamma}\left(\bar{\omega}\right)=\hat{\omega}+\frac{1}{\eta_2}\ln\left(\frac{b_i-b_r}{e^{\hat{\omega}}-b_r}\right)$$

Otherwise, if $\hat{\omega}<\ln b_i$, by construction $v\left(\hat{\omega},0;\hat{\omega},a(\hat{\omega}),\hat{\omega}\right)=\underline{v}$, meaning that $\underline{\Gamma}(\bar{\omega})\to\hat{\omega}$

as $\bar{\omega} \to a(\hat{\omega})$. Let us look at the behavior of $A\left(\underline{\omega}, \bar{\omega}, \hat{\omega}\right)$ as $\bar{\omega} \to +\infty$:

$$A\left(\underline{\omega}, \bar{\omega}, \hat{\omega}\right) \underset{\bar{\omega}\to+\infty}{=} (1-\eta_1)e^{(\eta_2-\eta_1)\underline{\omega}+(1-\eta_2)\hat{\omega}} + o(1)$$

Finally, we can also easily eastablish that:

$$\lim_{\bar{\omega}\to+\infty} \underline{\Gamma}\left(\bar{\omega}\right) = \hat{\omega} + \frac{1}{\eta_2}\ln\left(\frac{b_i - b_r}{\frac{\eta_2}{\eta_2-1}e^{\hat{\omega}} - b_r}\right)$$

Note that this last expression is less than $\hat{\omega}$ only if $\hat{\omega} > \ln\left(\frac{\eta_2-1}{\eta_2}b_i\right)$. We have thus established at the same time that there cannot be an equilibrium with "moderate minimum wage binding" when $\hat{\omega} < \ln\left(\frac{\eta_2-1}{\eta_2}b_i\right)$. □

Lemma 5. *The function $\hat{\omega} \to v\left(\hat{\omega}, 0; \underline{\Gamma}(\hat{\omega}), \hat{\omega}, \hat{\omega}\right)$ is convex and strictly increasing for $\hat{\omega} > \ln b_i$. It admits a minimum equal to $\underline{v}$ at $\hat{\omega} = \ln b_i$. $\hat{\omega}_* > \ln b_i$ is the unique point that satisfies $v\left(\hat{\omega}_*, 0; \underline{\Gamma}(\hat{\omega}_*), \hat{\omega}_*, \hat{\omega}_*\right) = \bar{v}$.*

We can use the analytical expression of v to express $v\left(\bar{\omega}, 0; \underline{\omega}, \bar{\omega}, \hat{\omega}\right)$ as follows:

$$v\left(\bar{\omega}, 0; \underline{\omega}, \bar{\omega}, \hat{\omega}\right) = \frac{e^{\bar{\omega}}}{\rho+\lambda-\mu-\frac{1}{2}\sigma^2} + \frac{\lambda\underline{v}}{\rho+\lambda} + \zeta_1 e^{\eta_1(\bar{\omega}-\hat{\omega})}\left(1 - \frac{\eta_1}{\eta_2}e^{(\eta_2-\eta_1)(\bar{\omega}-\underline{\omega})}\right) + c_1 e^{\eta_1(\bar{\omega}-\hat{\omega})} + a_1 e^{\eta_1\bar{\omega}} + a_2 e^{\eta_2\bar{\omega}}$$

Remember that ζ_1 is equal to $A(\underline{\omega}, \bar{\omega}, \hat{\omega})$ multiplied by a constant that is independent of $(\underline{\omega}, \bar{\omega})$. Remember also that $A(\underline{\omega}, \hat{\omega}, \hat{\omega}) = 0$. This means that $v\left(\hat{\omega}, 0; \underline{\omega}, \hat{\omega}, \hat{\omega}\right)$ has the following form:

$$v\left(\hat{\omega}, 0; \underline{\omega}, \hat{\omega}, \hat{\omega}\right) = \frac{e^{\hat{\omega}} + \lambda\underline{v}}{\rho+\lambda} - \frac{\eta_2}{\eta_2-\eta_1}\left(\frac{e^{\hat{\omega}} - b_r}{\rho+\lambda}\right)\left(1 - e^{-(\eta_2-\eta_1)(\hat{\omega}-\underline{\omega})}\right)$$

Finally, remember that when $\bar{\omega} \to \hat{\omega}$, we have $\underline{\Gamma}(\bar{\omega}) \to \hat{\omega} + \frac{1}{\eta_2}\ln\left(\frac{b_i-b_r}{e^{\hat{\omega}}-b_r}\right)$. We can reinject such expression into $\underline{\omega}$ to obtain:

$$v\left(\hat{\omega}, 0; \underline{\Gamma}(\hat{\omega}), \hat{\omega}, \hat{\omega}\right) = \left(\frac{b_i - b_r}{\rho+\lambda}\right)\left[\frac{b_r + \lambda\underline{v}}{b_i - b_r} + \frac{\eta_2}{\eta_2-\eta_1}\left(\frac{e^{\hat{\omega}} - b_r}{b_i - b_r}\right)^{\eta_1/\eta_2} - \frac{\eta_1}{\eta_2-\eta_1}\left(\frac{e^{\hat{\omega}} - b_r}{b_i - b_r}\right)\right]$$

Note that the function above is convex, and reaches a minimum when $\hat{\omega} = \ln b_i$. Such minimum is equal to $\underline{v}$ which is of course strictly less than $\bar{v}$. Thus, the equation $v(\hat{\omega}, 0; \underline{\Gamma}(\hat{\omega}), \hat{\omega}, \hat{\omega}) = \bar{v}$ admits a unique root $\hat{\omega}_* > \ln b_i$. $\square$

Lemma 6. If $\ln b_i < \hat{\omega} < \hat{\omega}_*$, there exists at least one equilibrium where $\underline{\omega} < \hat{\omega} < \bar{\omega}$.

To see this, focus on the function $\bar{\omega} \to v(\bar{\omega}, 0; \underline{\Gamma}(\bar{\omega}), \bar{\omega}, \hat{\omega})$. It is a continuous function of $\bar{\omega}$. When $\bar{\omega} = \hat{\omega}$, since $\hat{\omega} < \hat{\omega}_*$, using Lemma 5, we know that $v(\hat{\omega}, 0; \underline{\Gamma}(\hat{\omega}), \hat{\omega}, \hat{\omega}) < \bar{v}$. When $\bar{\omega} \to +\infty$, notice that $v(\bar{\omega}, 0; \underline{\Gamma}(\bar{\omega}), \bar{\omega}, \hat{\omega}) \to +\infty$. Thus, the intermediate value theorem can be applied, providing for the existence of at least one point $\bar{\omega}$ that satisfies:

$$v(\bar{\omega}, 0; \underline{\Gamma}(\bar{\omega}), \bar{\omega}, \hat{\omega}) = \bar{v}$$

Thus, $(\underline{\Gamma}(\bar{\omega}), \bar{\omega})$ is an equilibrium with moderate minimum wage. $\square$

Lemma 7. If $\ln b_i > \hat{\omega} >_* \hat{\omega}$, there exists at least one equilibrium where $\underline{\omega} < \hat{\omega} < \bar{\omega}$.

We are first going to prove that when $\ln b_i > \hat{\omega} >_* \hat{\omega}$,

$$v(a(\hat{\omega}), 0; \underline{\Gamma}(a(\hat{\omega})), a(\hat{\omega}), \hat{\omega}) = v(a(\hat{\omega}), 0; \hat{\omega}, a(\hat{\omega}), \hat{\omega}) < \bar{v}$$

Note that:

$$v(\bar{\omega}, 0; \hat{\omega}, \bar{\omega}, \hat{\omega}) = \frac{e^{\bar{\omega}}}{\rho + \lambda - \mu - \frac{1}{2}\sigma^2} + \frac{\lambda \underline{v}}{\rho + \lambda} + \zeta_1 e^{\eta_1(\bar{\omega} - \hat{\omega})} \left(1 - \frac{\eta_1}{\eta_2} e^{(\eta_2 - \eta_1)(\bar{\omega} - \hat{\omega})}\right) + a_1 e^{\eta_1 \bar{\omega}} + a_2 e^{\eta_2 \bar{\omega}}$$

This means:

$$v(\bar{\omega}, 0; \hat{\omega}, \bar{\omega}, \hat{\omega}) = \frac{1}{\rho + \lambda - \mu - \frac{1}{2}\sigma^2} \left[e^{\bar{\omega}} + \frac{e^{\eta_1 \bar{\omega}} \left(\eta_2 - \eta_1 e^{(\eta_2 - \eta_1)(\bar{\omega} - \hat{\omega})}\right) A(\hat{\omega}, \bar{\omega}, \hat{\omega})}{(-\eta_1 \eta_2)(\eta_2 - \eta_1)} + \frac{\eta_2(\eta_2 - 1) e^{\eta_1(\bar{\omega} - \hat{\omega})} + \eta_1(1 - \eta_1) e^{\eta_2(\bar{\omega} - \hat{\omega})}}{(-\eta_1 \eta_2)(\eta_2 - \eta_1)} e^{\hat{\omega}} \right] + \frac{\lambda \underline{v}}{\rho + \lambda}$$

Then note that $A(\hat{\omega}, a(\hat{\omega}), \hat{\omega}) = (1 - \eta_1)(\eta_2 - 1) e^{-\eta_1 \hat{\omega}} (b_i - e^{\hat{\omega}})$. This means that when $\bar{\omega} = a(\hat{\omega})$, we have:

$$v\left(a(\hat{\omega}),0;\hat{\omega},a(\hat{\omega}),\hat{\omega}\right)=\frac{\lambda\underline{v}}{\rho+\lambda}$$
$$+\frac{\left[\eta_2(1-\eta_1)e^{\eta_1(a(\hat{\omega})-\hat{\omega})}+\eta_1(\eta_2-1)e^{\eta_2(a(\hat{\omega})-\hat{\omega})}\right]b_i+(-\eta_1\eta_2)\left[e^{\eta_2(a(\hat{\omega})-\hat{\omega})}-e^{\eta_1(a(\hat{\omega})-\hat{\omega})}\right]e^{\hat{\omega}}}{(\eta_2-\eta_1)(\rho+\lambda)}$$

Note that this last expression is equal to $\underline{v}$ when $\hat{\omega}=\ln b_i$. For $\ln\left(\frac{\eta_2-1}{\eta_2}b_i\right)<\hat{\omega}<\ln b_i$, some algebra can show that the function $\hat{\omega}\to v\left(a(\hat{\omega}),0;\hat{\omega},a(\hat{\omega}),\hat{\omega}\right)$ is decreasing – this is obvious if one realizes that this function is identical to the function $\underline{\omega}\to v\left(a(\underline{\omega});\underline{\omega},a(\underline{\omega})\right)$ where v is the value function in the absence of minimum wage (see (A26)), and for which Alvarez and Shimer (2011) has proven the monotonicity. Note also that $\hat{\omega}\to v\left(a(\hat{\omega}),0;\hat{\omega},a(\hat{\omega}),\hat{\omega}\right)$ has limit $+\infty$ when $\hat{\omega}\to\ln\left(\frac{\eta_2-1}{\eta_2}b_i\right)$ and limit $\underline{v}$ when $\hat{\omega}\to\ln b_i$. Finally, ${}_*\hat{\omega}$ is the unique point that satisfies $v\left(a({}_*\hat{\omega}),0;,_*\hat{\omega},a({}_*\hat{\omega}),_*\hat{\omega}\right)=\bar{v}$. In other words, we have demonstrated that so long as $\hat{\omega}\in({}_*\hat{\omega},\ln b_i)$, we have:

$$v\left(a(\hat{\omega}),0;\underline{\Gamma}\left(a(\hat{\omega})\right),a(\hat{\omega}),\hat{\omega}\right)<\bar{v}$$

Since the function $\bar{\omega}\to v\left(\bar{\omega},0;\underline{\Gamma}\left(\bar{\omega}\right),\bar{\omega},\hat{\omega}\right)$ is continuous on $[a(\hat{\omega});+\infty]$, with value strictly less than $\bar{v}$ at $\bar{\omega}=a(\hat{\omega})$ and with limit $+\infty$ as $\bar{\omega}\to+\infty$, the intermediate value theorem provides for the existence of $\bar{\omega}>a(\hat{\omega})$ such that $v\left(\bar{\omega},0;\underline{\Gamma}(\bar{\omega}),\bar{\omega},\hat{\omega}\right)=\bar{v}$. □

A.2.7.3 Low Minimum Wage

The existence and uniqueness of a non-binding minimum wage equilibrium is established in Alvarez and Shimer (2011), whenever $\hat{\omega}<_*\hat{\omega}$, and the proof is thus omitted. □

A.2.8 Hazard Rate of Exiting Unemployment

A.2.8.1 Irrelevance of Seniority when Entering Unemployment Spell

Consider a worker with seniority s who is rest unemployed whenever $s<1-e^{\theta(\omega-\hat{\omega})}$. Using the definition of ω in (36) and suppressing the dependence of these variables on the labor market and minimum wage, we can write this as a condition relating the number of more senior workers in the market, $\ell(1-s)$, to the current productivity of the market x_0,

$$\ell(1-s) > \left(\frac{Ax_0}{C}\right)^{\theta-1} e^{-\theta\hat{\omega}}.$$

The worker exits rest unemployment and returns to this market the next time this inequality is violated, i.e. when productivity reaches $\hat{x}$ solving

$$\ell(1-s) = \left(\frac{A\hat{x}}{C}\right)^{\theta-1} e^{-\theta\hat{\omega}}.$$

Conversely, she exits rest unemployment and leaves the market when she first reaches state $(\underline{\omega}, 0)$, which occurs at the productivity level $\underline{x}$ satisfying

$$\ell(1-s) = \left(\frac{A\underline{x}}{C}\right)^{\theta-1} e^{-\theta\underline{\omega}},$$

so the log full employment wage is $\underline{\omega}$ if there are $\ell(1-s)$ workers left in the market. She also exits the market exogenously if she quits or the market breaks down, at rate $\lambda = q+\delta$. Thus the hazard of ending a spell of rest unemployment depends on competing hazards of productivity rising to $\hat{x}$ or falling to $\underline{x}$. The key observation is that the ratio of these two thresholds is monotone in the distance between $\hat{\omega}$ and $\underline{\omega}$,

$$\hat{\omega} - \underline{\omega} = \frac{\theta-1}{\theta}\left(\ln\hat{x} - \ln\underline{x}\right),$$

and so is the same for all workers in a labor market, regardless of their seniority. □

A.2.8.2 Hazard Rate of Exiting Unemployment

Denote $u_r(t)$ (resp. $u_s(t)$) the duration-contingent rest (resp. search) unemployment probability; the hazard of ending a (rest or search) unemployment spell of duration t, $h(t)$, is equal to:

$$h(t) = \hat{h}_r(t)\frac{u_r(t)}{u_r(t)+u_s(t)} + \alpha\frac{u_s(t)}{u_r(t)+u_s(t)},$$

where $\frac{u_r(t)}{u_r(t)+u_s(t)}$ is the probability that a worker with unemployment duration t is rest-unemployed. For a search-unemployed worker, spells end at rate α, independent of the duration of the spell. For a rest-unemployed worker, her spell ends when local labor market conditions improve enough for her to reenter employment. The duration-contingent unemployment rates solve a system of two ordinary differential equations with time-varying coefficients:

$$\dot{u}_r(t) = -u_r(t)(\delta + q + \underline{h}_r(t) + \hat{h}_r(t)) \text{ and } \dot{u}_s(t) = -u_s(t)\alpha + u_r(t)(\delta + q + \underline{h}_r(t)) \quad \text{(A40)}$$

for all $t > 0$. The number of workers in rest unemployment falls as markets shut down and workers exogenously quit, as they exit the market for search unemployment, and as they reenter employment. In the first three events, they become search unemployed, while search unemployment falls at rate α as these workers find jobs. To solve these differential equations, we require two boundary conditions; however, to compute the share of rest unemployed in the unemployed population with duration t, $\frac{u_r(t)}{u_r(t)+u_s(t)}$, we need only a single boundary condition,

$$\frac{\int_0^\infty u_r(t)dt}{\int_0^\infty u_s(t)dt} = \frac{U_r}{U_s}, \quad \text{(A41)}$$

where U_r and U_s are given in equations (55) and (57). In other words:

$$u_r(t) = u_r(0) \exp\left[-\int_0^t \left(\delta + q + \underline{h}_r(s) + \hat{h}_r(s)\right) ds\right]$$
$$u_s(t) = e^{-\alpha t}u_s(0) + \int_0^t e^{-\alpha(t-s)}u_r(s)\left(\delta + q + \underline{h}_r(s)\right) ds$$

□

A.2.8.3 Hazard Rate Computations

Consider a Brownian motion with initial $\omega \in (\underline{\omega}, \hat{\omega})$. Denote $\psi_m \equiv \frac{1}{2}\left(\frac{\mu^2}{\sigma^2} + \frac{m^2\pi^2\sigma^2}{(\hat{\omega}-\underline{\omega})^2}\right)$ the eigen-values of the infinitessimal operator for $\omega(t)$. Let $\hat{G}(t;\cdot;\cdot)$ and $\underline{G}(t;\cdot;\cdot)$ denote the cumulative distribution function for the times until each of the barriers is hit, conditional on the initial value of ω:

$$\hat{G}(t;\hat{\omega},\underline{\omega};\omega) = \Pr\{T_{\hat{\omega}} \le t, T_{\hat{\omega}} < T_{\underline{\omega}} \,|\omega(0) = \omega\}$$
$$\underline{G}(t;\hat{\omega},\underline{\omega};\omega) = \Pr\{T_{\underline{\omega}} \le t, T_{\underline{\omega}} < T_{\hat{\omega}} \,|\omega(0) = \omega\},$$

with associated densities $\hat{g}$ and $\underline{g}$. Adam W. Kolkiewicz (2002, pp. 17–18) proves

$$\hat{g}(t;\hat{\omega},\underline{\omega};\omega) = \frac{\pi\sigma^2}{(\hat{\omega}-\underline{\omega})^2}\sum_{n=1}^{\infty} n(-1)^{n-1}\sin\left(\frac{\pi n(\omega-\underline{\omega})}{\hat{\omega}-\underline{\omega}}\right)e^{\frac{\mu(2(\hat{\omega}-\omega)-\mu t)}{2\sigma^2}-\frac{\pi^2 n^2\sigma^2 t}{2(\hat{\omega}-\underline{\omega})^2}}$$

$$\underline{g}(t;\hat{\omega},\underline{\omega};\omega) = \frac{\pi\sigma^2}{(\hat{\omega}-\underline{\omega})^2}\sum_{n=1}^{\infty} n(-1)^{n-1}\sin\left(\frac{\pi n(\hat{\omega}-\omega)}{\hat{\omega}-\underline{\omega}}\right)e^{\frac{-\mu(2(\omega-\underline{\omega})+\mu t)}{2\sigma^2}-\frac{\pi^2 n^2\sigma^2 t}{2(\hat{\omega}-\underline{\omega})^2}}.$$

The hazard rate of the first hitting time, conditional on a rest unemployment spell starting at time 0, i.e conditional on $\omega=\hat{\omega}$, is

$$\hat{h}_r(t) \equiv \lim_{\omega\uparrow\hat{\omega}} \frac{\hat{g}(t;\hat{\omega},\underline{\omega},\omega)}{1-\hat{G}(t;\hat{\omega},\underline{\omega},\omega)-\underline{G}(t;\hat{\omega},\underline{\omega},\omega)} \text{ and } \underline{h}_r(t) \equiv \lim_{\omega\uparrow\hat{\omega}} \frac{\underline{g}(t;\hat{\omega},\underline{\omega},\omega)}{1-\hat{G}(t;\hat{\omega},\underline{\omega},\omega)-\underline{G}(t;\hat{\omega},\underline{\omega},\omega)}.$$

We compute this hazard rate in two steps. First, note that the survival distribution function verifies:

$$\begin{aligned}
&1-\hat{G}(t;\hat{\omega},\underline{\omega};\omega)-\underline{G}(t;\hat{\omega},\underline{\omega};\omega) = \int_t^{+\infty}\Big(\hat{g}(s;\hat{\omega},\underline{\omega};\omega)+\underline{g}(s;\hat{\omega},\underline{\omega};\omega)\Big)\,ds \\
&= \frac{\pi\sigma^2}{(\hat{\omega}-\underline{\omega})^2}\sum_{n=1}^{\infty} n(-1)^{n-1}\frac{e^{-\psi_n t}}{\psi_n}\left(e^{\frac{\mu(\hat{\omega}-\omega)}{\sigma^2}}\sin\left(\frac{\pi n(\omega-\underline{\omega})}{\hat{\omega}-\underline{\omega}}\right)+e^{\frac{-\mu(\omega-\underline{\omega})}{\sigma^2}}\sin\left(\frac{\pi n(\hat{\omega}-\omega)}{\hat{\omega}-\underline{\omega}}\right)\right)
\end{aligned}$$

This is the case since for $t>0$, Fubini's theorem applies on the interval $[t,+\infty)$. Indeed, focusing on $\int_t^{+\infty}\hat{g}(s;\hat{\omega},\underline{\omega};\omega)ds$ for example, and for a fixed integer N, we have the following bound:

$$\begin{aligned}
&\frac{\pi\sigma^2}{(\hat{\omega}-\underline{\omega})^2}\int_t^{+\infty}\sum_{n=1}^{N} n\left|\sin\left(\frac{\pi n(\omega-\underline{\omega})}{\hat{\omega}-\underline{\omega}}\right)\right| e^{\frac{\mu(2(\hat{\omega}-\omega)-\mu s)}{2\sigma^2}-\frac{\pi^2 n^2\sigma^2 s}{2(\hat{\omega}-\underline{\omega})^2}}ds \\
&\qquad = \frac{\pi\sigma^2}{(\hat{\omega}-\underline{\omega})^2}\sum_{n=1}^{N}\frac{ne^{-\psi_n t}}{\psi_n}e^{\frac{\mu(\hat{\omega}-\omega)}{\sigma^2}}\left|\sin\left(\frac{\pi n(\omega-\underline{\omega})}{\hat{\omega}-\underline{\omega}}\right)\right| \\
&\qquad \le \frac{\pi\sigma^2}{(\hat{\omega}-\underline{\omega})^2}\sum_{n=1}^{\infty}\frac{ne^{-\psi_n t}}{\psi_n}e^{\frac{\mu(\hat{\omega}-\omega)}{\sigma^2}} < +\infty
\end{aligned}$$

The last inequality is due to the fact that $\psi_n \sim n^2$ and therefore the term $e^{-\psi_n t}$ guarantees the convergence of the series when $t>0$. A similar reasoning applies to $\int_t^{+\infty}\underline{g}(s;\hat{\omega},\underline{\omega};\omega)ds$.

Using L'Hopital's rule, and noting that:

$$-\left[\hat{G}_\omega(t;\hat{\omega},\underline{\omega};\hat{\omega})+\underline{G}_\omega(t;\hat{\omega},\underline{\omega};\hat{\omega})\right]=-\frac{\pi^2\sigma^2}{(\hat{\omega}-\underline{\omega})^3}\sum_{n=1}^{\infty}n^2\frac{e^{-\psi_n t}}{\psi_n}\left(1-(-1)^n e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}\right)$$

and that:

$$\hat{g}_\omega(t;\hat{\omega},\underline{\omega};\hat{\omega})=-\frac{\pi^2\sigma^2}{(\hat{\omega}-\underline{\omega})^3}\sum_{n=1}^{\infty}n^2e^{-\psi_n t}$$

$$\underline{g}_\omega(t;\hat{\omega},\underline{\omega};\hat{\omega})=\frac{\pi^2\sigma^2e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}}{(\hat{\omega}-\underline{\omega})^3}\sum_{n=1}^{\infty}(-1)^n n^2e^{-\psi_n t},$$

one obtains the hitting time hazard rates $\hat{h}_r(t)$ and $\underline{h}_r(t)$:

$$\hat{h}_r(t)=\frac{\sum_{m=1}^{\infty}m^2e^{-t\psi_m}}{\sum_{m=1}^{\infty}\frac{m^2}{\psi_m}e^{-t\psi_m}\left(1-(-1)^m e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}\right)} \tag{A42}$$

$$\underline{h}_r(t)=\frac{-\sum_{m=1}^{\infty}m^2e^{-t\psi_m}(-1)^m e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}}{\sum_{m=1}^{\infty}\frac{m^2}{\psi_m}e^{-t\psi_m}\left(1-(-1)^m e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}\right)}. \tag{A43}$$

□

A.2.8.4 Hazard Rate Limits

The hazard rate is particularly easy to characterize both at short and long durations. From (A42) it can be seen that $\lim_{t\to 0}h(t)t=1/2$. Alternatively, when t is small, we find that $\hat{h}_r(t)\approx 1/(2t)$.

When t is large, the first term of the partial sum in (A42) dominates,

$$\lim_{t\to\infty}\hat{h}_r(t)=\frac{\psi_1}{1+e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}}\text{ and }\lim_{t\to\infty}\underline{h}_r(t)=\frac{\psi_1e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}}{1+e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}}.$$

In addition, if $\alpha > \delta + q + \psi_1$,

$$\lim_{t\to\infty} \frac{u_r(t)}{u_s(t)} = \frac{(\alpha - \psi_1 - \delta - q)\left(1 + e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}\right)}{\delta + q + (\delta + q + \psi_1)e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}},$$

while otherwise the limiting ratio is zero. Together this implies $\lim_{t\to\infty} h(t) = \min\{\alpha, \psi_1 + \delta + q\}$, a function only of the slower exit rate. □

A.2.9 Random Allocation Model

It is straightforward to show that v satisfies the following differential equation, for $\omega \in [\underline{\omega}_r, \bar{\omega}_r]$ and $\omega \neq \hat{\omega}$:

$$(\rho + \lambda)\, v(\omega; \hat{\omega}, \underline{\omega}_r, \bar{\omega}_r) = R(\omega) + \lambda \underline{v} + \mu v_\omega(\omega; \hat{\omega}, \underline{\omega}_r, \bar{\omega}_r) + \tfrac{1}{2}\sigma^2 v_{\omega\omega}(\omega; \hat{\omega}, \underline{\omega}_r, \bar{\omega}_r)$$

Moreover, v is continuously differentiable at $\omega = \hat{\omega}$ and must satisfy the boundary conditions $v_\omega(\underline{\omega}_r; \hat{\omega}, \underline{\omega}_r, \bar{\omega}_r) = v_\omega(\bar{\omega}_r; \hat{\omega}, \underline{\omega}_r, \bar{\omega}_r) = 0$.

A.2.9.1 Intermediate minimum wage

In the case where $\underline{\omega}_r < \hat{\omega} < \bar{\omega}_r$, the solution to the HJB equation takes the following form

$$v(\omega; \hat{\omega}, \underline{\omega}_r, \bar{\omega}_r) = \begin{cases} \dfrac{e^{\omega}}{\rho + \lambda - \mu - \frac{1}{2}\sigma^2} + \dfrac{\lambda \underline{v}}{\rho + \lambda} + \bar{c}_1 e^{\eta_1(\omega - \hat{\omega})} + \bar{c}_2 e^{\eta_2(\omega - \hat{\omega})} & \text{if } \omega \geq \hat{\omega} \\ \dfrac{\left(e^{\hat{\omega}} - b_r\right) e^{\theta(\omega - \hat{\omega})}}{\rho + \lambda - \mu\theta - \frac{1}{2}(\sigma\theta)^2} + \dfrac{b_r + \lambda \underline{v}}{\rho + \lambda} + \underline{c}_1 e^{\eta_1(\omega - \hat{\omega})} + \underline{c}_2 e^{\eta_2(\omega - \hat{\omega})} & \text{if } \omega < \hat{\omega}, \end{cases} \tag{A44}$$

where $\eta_1 < 0 < 1 < \eta_2$ solve (40) and the constants $\bar{c}_1, \bar{c}_2, \underline{c}_1, \underline{c}_1$ satisfy the boundary conditions and the requirement that v be continuously differentiable at $\hat{\omega}$:

$$\frac{\theta\left(e^{\hat{\omega}} - b_r\right) e^{\theta(\underline{\omega}_r - \hat{\omega})_r}}{\rho + \lambda - \mu\theta - \frac{1}{2}(\sigma\theta)^2} + \underline{c}_1 \eta_1 e^{\eta_1(\underline{\omega}_r - \hat{\omega})} + \underline{c}_2 \eta_2 e^{\eta_2(\underline{\omega}_r - \hat{\omega})} = 0$$

$$\frac{e^{\bar{\omega}_r}}{\rho + \lambda - \mu - \frac{1}{2}\sigma^2} + \bar{c}_1 \eta_1 e^{\eta_1(\bar{\omega}_r - \hat{\omega})} + \bar{c}_2 \eta_2 e^{\eta_2(\bar{\omega}_r - \hat{\omega})} = 0$$

$$\frac{e^{\hat{\omega}}}{\rho+\lambda-\mu-\frac{1}{2}\sigma^2}+\frac{\lambda \underline{v}}{\rho+\lambda}+\bar{c}_1+\bar{c}_2=\frac{\left(e^{\hat{\omega}}-b_r\right)}{\rho+\lambda-\mu\theta-\frac{1}{2}(\sigma\theta)^2}+\frac{b_r+\lambda\underline{v}}{\rho+\lambda}+\underline{c}_1+\underline{c}_2$$

$$\frac{e^{\hat{\omega}}}{\rho+\lambda-\mu-\frac{1}{2}\sigma^2}+\eta_1\bar{c}_1+\eta_2\bar{c}_2=\frac{\left(e^{\hat{\omega}}-b_r\right)\theta}{\rho+\lambda-\mu\theta-\frac{1}{2}(\sigma\theta)^2}+\eta_1\underline{c}_1+\eta_2\underline{c}_2$$

One can alternatively show that the value function can be expressed with the use of the discounted occupancy function Π, defined via:

$$\Pi\left(\omega';\omega;\hat{\omega},\underline{\omega}_r,\bar{\omega}_r\right)\equiv\mathbb{E}_\omega\left[\int_0^{+\infty}e^{-(\rho+\lambda)t}I_{\omega'}\left(\omega(t)\right)dt\right],$$

with $I_{\omega'}(\omega)$ the indicator function equal to 1 when $\omega\leq\omega'$ and zero otherwise. Arguments similar to those in Alvarez and Shimer (2011) then show that

$$v(\omega;\hat{\omega},\underline{\omega}_r,\bar{\omega}_r)=\int_{\underline{\omega}_r}^{\bar{\omega}_r}\left[\min\left(e^{\omega},(1-u(\omega))\,e^{\hat{\omega}}+u(\omega)b_r\right)+\lambda\underline{v}\right]\Pi_{\omega'}\left(\omega';\omega;\hat{\omega},\underline{\omega}_r,\bar{\omega}_r\right)d\omega',$$

where the discounted local time function $\Pi_{\omega'}\left(\omega';\omega;\hat{\omega},\underline{\omega}_r,\bar{\omega}_r\right)$ is only dependent upon the roots η_1,η_2, as well as the barriers $\hat{\omega},\underline{\omega}_r,\bar{\omega}_r$ and the effective discount rate $\rho+\lambda$ (see Stokey (2008)). Arguments similar to those used in Alvarez and Shimer (2011), and the fact that the period return function $\min\left(e^{\omega},(1-u(\omega))\,e^{\hat{\omega}}+u(\omega)b_r\right)$ is monotone increasing in ω, can then show that v is increasing in the arguments $\omega,\underline{\omega}_r,\bar{\omega}_r$, and strictly so when $\underline{\omega}_r<\omega<\bar{\omega}_r$.

A.2.9.2 High minimum wage

In the case where $\hat{\omega}>\bar{\omega}_r$, the solution to the HJB equation takes the following form:

$$v(\omega;\hat{\omega},\underline{\omega}_r,\bar{\omega}_r)=\frac{\left(e^{\hat{\omega}}-b_r\right)e^{\theta(\omega-\hat{\omega})}}{\rho+\lambda-\mu\theta-\frac{1}{2}(\sigma\theta)^2}+\frac{b_r+\lambda\underline{v}}{\rho+\lambda}+c_1e^{\eta_1(\omega-\hat{\omega})}+c_2e^{\eta_2(\omega-\hat{\omega})}\tag{A45}$$

where the constants c_1,c_2 satisfy the boundary conditions:

$$\frac{\theta\left(e^{\hat{\omega}}-b_r\right)e^{\theta(\underline{\omega}_r-\hat{\omega})}}{\rho+\lambda-\mu\theta-\frac{1}{2}(\sigma\theta)^2}+c_1\eta_1e^{\eta_1(\underline{\omega}_r-\hat{\omega})}+c_2\eta_2e^{\eta_2(\underline{\omega}_r-\hat{\omega})}=0$$

$$\frac{\theta\left(e^{\hat{\omega}}-b_r\right)e^{\theta(\bar{\omega}_r-\hat{\omega})}}{\rho+\lambda-\mu\theta-\frac{1}{2}(\sigma\theta)^2}+c_1\eta_1e^{\eta_1(\bar{\omega}_r-\hat{\omega})}+c_2\eta_2e^{\eta_2(\bar{\omega}_r-\hat{\omega})}=0$$

Some algebra delivers

$$c_1 = \left(\frac{\theta}{\eta_1}\right)\left(\frac{e^{\hat{\omega}} - b_r}{\frac{1}{2}(\sigma\theta)^2 + \mu\theta - (\rho+\lambda)}\right)\left(\frac{e^{\eta_2(\bar{\omega}_r - \hat{\omega}) + \theta(\underline{\omega}_r - \hat{\omega})} - e^{\eta_2(\underline{\omega}_r - \hat{\omega}) + \theta(\bar{\omega}_r - \hat{\omega})}}{e^{\eta_1(\underline{\omega}_r - \hat{\omega}) + \eta_2(\bar{\omega}_r - \hat{\omega})} - e^{\eta_1(\bar{\omega}_r - \hat{\omega}) + \eta_2(\underline{\omega}_r - \hat{\omega})}}\right)$$

$$c_2 = \left(\frac{\theta}{\eta_2}\right)\left(\frac{e^{\hat{\omega}} - b_r}{\frac{1}{2}(\sigma\theta)^2 + \mu\theta - (\rho+\lambda)}\right)\left(\frac{e^{\eta_1(\underline{\omega}_r - \hat{\omega}) + \theta(\bar{\omega}_r - \hat{\omega})} - e^{\eta_1(\bar{\omega}_r - \hat{\omega}) + \theta(\underline{\omega}_r - \hat{\omega})}}{e^{\eta_1(\underline{\omega}_r - \hat{\omega}) + \eta_2(\bar{\omega}_r - \hat{\omega})} - e^{\eta_1(\bar{\omega}_r - \hat{\omega}) + \eta_2(\underline{\omega}_r - \hat{\omega})}}\right)$$